MONTHLY NOTICES

OF THE

ROYAL ASTRONOMICAL SOCIETY.

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Dr. H. KNOX-SHAW, PRESIDENT, in the chair.

The election by the Council of the following Fellows was duly confirmed :—

Paul Adrien Maurice Dirac, Ph.D., F.R.S., St. John's College, Cambridge ;

Captain James Ritchie Frost, Principal Dockmaster, Tilbury Docks, 2 The Gardens, Tilbury, Essex ;

John Watson Maccoll, B.Sc., Ph.D., D.I.C., Ballistic Office, Ordnance Committee, Woolwich, London, S.E. 18 ;

G. C. McVittie, M.A., Ph.D., Department of Mathematics, The University, Leeds ;

W. H. Owston, Heath View Hotel, 15 and 16 Montpelier Row, Blackheath, London, S.E. 3 ;

Abdul-Hamid Mahmoud Samaha, Assistant Astronomer, Helwan Observatory, Egypt ; and

Joe Ueta, D.Sc., Assistant Professor of Astronomy, Kyoto Imperial University, Kyoto University Observatory, Kyoto, Japan.

The following candidates were proposed for election as Fellows of the Society, the names of the proposers from personal knowledge being appended :—

John Robinson Airey, M.A., Sc.D., D.Sc., Queenswood, Beckett Park, Leeds (proposed by L. J. Comrie) ;

Arthur Coombe, M.A., Headmaster of Chichester School, Chichester (proposed by G. Street) ;

Rev. Henry Norman Edge, 11 Groombridge Road, South Hackney, London, E. 9 (proposed by W. J. S. Lockyer) ;

Henry Patrick Folkard, Curraghmore, Edgewarebury Lane, Edgeware, Middlesex (proposed by W. A. Parr) ;
Vice-Admiral C. A. Fountaine, C.B., Narford Hall, King's Lynn, Norfolk (proposed by R. A. Sampson) ;
Miss Colleen B. Orpen, B.Sc., P.O., Afrika's Kop, O.F.S., S. Africa (proposed by H. Spencer Jones) ; and
Laurence Chisholm Young, B.A., Trinity College, Cambridge (proposed by J. C. P. Miller).

Sixty-nine presents were announced as having been received since the last meeting, including

Canberra, Memoirs of the Commonwealth Solar Observatory, No. 2.
Gold Pocket Chronometer (Padbury 3474) in mahogany case (presented by Mr. J. J. Hall).

The President announced that Professor W. de Sitter would deliver the George Darwin Lecture at the Ordinary Meeting of the Society on May 8, taking as his subject " Jupiter's Satellites."

The Parallax of Nova Pictoris. By H. Spencer Jones, F.R.S., H.M. Astronomer at the Cape.

Nova Pictoris was placed on the parallax observing programme with the Victoria 24-inch telescope in 1928. Observations were obtained at 5 epochs, the magnitude of the nova being reduced at each epoch to approximate equality with the magnitudes of the comparison stars by means of the rotating sector.

The following is a summary of the observations :—

Plate No.	Date.	Hour Angle (min.).	Time Factor (yrs.).	Parallax Factor.	Weight.	Solution (Unit $0^{s}\cdot0001$).	Residual (Unit $0^{s}\cdot0001$).
5108	1928 Nov. 4	−10·7	−1·157	+·826	1·00	−169	+ 6
5285	28	+14·6	−1·091	+·528	1·00	−198	−24
5317	Dec. 4	− 3·1	−1·074	+·437	1·00	−189	−15
5333	5	+ 7·3	−1·072	+·421	1·00	−141	+32
5609	1929 Jan. 30	+ 7·1	−0·918	−·529	0·60	−172	− 2
5632	Feb. 1	−15·0	−0·913	−·558	0·60	−169	0
5750	18	− 9·5	−0·866	−·774	0·60	−141	+28
5864	Mar. 13	+ 3·1	−0·803	−·957	0·60	−139	+30
5915	20	+17·1	−0·784	−·984	1·00	−181	−12
5934	25	+33·1	−0·771	−·994	0·60	−181	−11
7064	Oct. 30	−16·3	−0·173	+·873	0·60	−200	0
7103	Nov. 6	− 4·0	−0·154	+·808	0·60	−238	−38

Plate No.	Date.	Hour Angle (min.).	Time Factor (yrs.).	Parallax Factor.	Weight.	Solution (Unit 0r.0001).	Residual (Unit 0r.0001).
7191	1929 Dec. 2	+10·8	−0·083	+·471	0·50	−151	+48
7205	5	+17·9	−0·074	+·424	0·50	−214	−16
7504	1930 Feb. 3	−13·0	+0·093	−·583	0·60	−197	− 3
7644	27	+ 5·2	+0·158	−·860	0·80	−207	−13
7727	Mar. 12	+ 3·1	+0·194	−·953	0·35	−207	−13
8987	Oct. 29	− 7·0	+0·824	+·884	0·80	−199	+26
9016	Nov. 1	− 1·4	+0·833	+·858	1·00	−237	−12
9025	2	+ 4·9	+0·835	+·849	0·80	−230	− 5
9031	3	+ 2·0	+0·838	+·840	0·80	−216	+ 9

The normal equations are

$$\begin{aligned} 15{\cdot}35c - 4{\cdot}937\mu + 1{\cdot}943\pi &= -0^{r}{\cdot}2914 \\ + 10{\cdot}078 + 2{\cdot}258 &= +0^{r}{\cdot}0699 \\ + 8{\cdot}817 &= -0^{r}{\cdot}0512 \end{aligned}$$

giving

$$\begin{aligned} c &= -0^{r}{\cdot}0197 \\ \mu &= -0^{r}{\cdot}0025 = -0''{\cdot}039 \pm 0''{\cdot}007 \\ \pi &= -0^{r}{\cdot}0008 = -0''{\cdot}013 \pm 0''{\cdot}007. \end{aligned}$$

The probable error for unit weight is $\pm 0''{\cdot}019$.

The relative proper motion in R.A. is in close agreement with the proper motion of $-0''{\cdot}042 \pm 0''{\cdot}013$ derived at Harvard * from the comparison of two pairs of plates obtained with the 24-inch Bruce refractor, with an average interval of 24·2 years. The later plates were obtained after the outburst, the brightness of the nova being reduced by means of a diffraction grating.

The mean parallax of the comparison stars may be taken as $0''{\cdot}004$. The absolute parallax of the nova, as determined trigonometrically, is thus

$$-0''{\cdot}009 \pm 0''{\cdot}007.$$

The spectroscopic evidence indicates that the spectrum immediately after discovery was of the super-giant type, indicating high absolute luminosity. The direct trigonometrical determination of the parallax merely confirms this. Davidovich † has estimated the absolute magnitude at maximum brightness as − 5·0 by comparing the relative intensities of selected pairs of lines in the spectrum of ι' Scorpii and in the spectra of the nova on May 28 and June 8. The corresponding parallax is $0''{\cdot}006$. The estimate is an extrapolation, and it will be shown that the parallax derived in this way is too great.

The hypothesis will be adopted that the main phenomena exhibited by the spectrum of the nova subsequent to the outburst are to be attributed to a shell or a series of shells of gas thrown off from the nova

* *H.C.O. Bulletin*, No. 852, 1927.
† *H.C.O. Circular*, No. 295, 1926.

as a result of the outburst. This hypothesis, which has been described by Evershed * among others, has gained wide acceptance in recent years and has received theoretical support from the investigations of Milne.† The observations of Nova Pictoris at the Cape Observatory lend support to the hypothesis as follows :—

(1) The absorption lines should be situated at the short wave-length edges of the emission bands. In all cases where absorption and emission are associated in Nova Pictoris this requirement is fulfilled. Further, where two or three absorptions and emissions of the same line occur simultaneously the emissions are superposed, giving a complex emission band with the appropriate absorption line at the corresponding edge of each emission.

(2) The centre of each emission band should have a velocity displacement which corresponds to the actual line of sight velocity of the nova. In consequence, where two or more emissions of the same wave-length are superposed they should have a common centre. The measurements of the displacements of the centres of the emission bands made at various stages are as follows :—

Plate Nos.	Dates.	No. of Plates.	Mean Disp. of Centres in km./sec.
7320–26	1925 June 12, 17	2	+14·9
7340–50	„ 25, 29	2	+10·7
7407–15	Aug. 28–Sept. 10	4	+12·5
7465–7501	1925 Dec. 3–1926 Mar. 13	21	+12·2

The measured displacements agree within the limits of observational error and indicate a radial velocity of about + 12 km./sec.

No measurements were made of the centres of superposed emissions in the case of complex emission bands, as the weak broader emissions have very diffuse edges and accurate measures are not possible. It is apparent, however, from an inspection of the spectrograms that the superposed emissions have approximately coincident centres, as required by theory. It is also noticeable that as the displacement towards the violet of an absorption line increases, the emission not only moves towards the violet with the absorption line, but that it also moves towards the red by an equal amount.

(3) The widths of the bands are due to the Doppler effect and should therefore be proportional to the wave-lengths. This relationship cannot be decisively tested by the measures of the Cape 4-prism spectrograms owing to the limited spectral region available (approximately λ 3800 – λ 5000), but it appears to be reasonably well satisfied.

According to Milne's theory, the ultimate velocity of the ejected matter will be of the order of several times the velocity of escape from the star under the action of gravity alone. The velocity of escape from the Sun is 622 km./sec. and within the observed range of stellar masses it does not vary greatly for dwarf stars. In general, the displacements observed in the spectra of novæ are large and correspond to velocities of the order of a few thousand kilometres per second, in good

* *M.N.R.A.S.*, **79**, 482, 1919. † *Ibid.*, **86**, 459, 1926.

qualitative agreement with Milne's estimate. In the case of Nova Pictoris, on the other hand, large velocity displacements were only observed occasionally, and the absorptions which were associated with these large displacements were short-lived. The main absorption and emission were associated throughout the period of observation with a velocity which was small relative to the velocities observed in previous novæ. The velocity displacement for the first few days of observation (until the first maximum was attained) was about 65 km./sec. ; it then increased to a velocity of the order of 300 km./sec., remaining at about this value for four months. A further increase to a velocity of about 400 km./sec. then took place, the velocity remaining at this figure until the conclusion of the series of observations six months later. The velocity displacement up to the time of the first observed maximum was so low that there seems to be no escape from the conclusion that the ejection of a gaseous shell did not occur in the initial stages of the outburst, but that the observed increase in brightness was due to a general swelling or expansion of the outer layers of the star. Numerical estimates of the rate of expansion will now be considered.

If m_0 and m are the bolometric magnitudes of the star before the outbreak, and after the outbreak but before the first maximum respectively, R_0, T_0 and R, T, the corresponding radii and temperatures, the following relation must hold :—

$$m_0 - m = 5 \log RT^2/R_0T_0^2.$$

Within the range of temperatures with which we are concerned (around 6000° C.) the corrections to reduce bolometric magnitudes to visual magnitudes are practically negligible and well within the probable errors of the magnitude determinations. It will be sufficiently accurate, therefore, to employ visual magnitudes in this formula.

The effective temperature of the nova near maximum will be assumed to be 5700°. It is well known that the effective temperature of a giant star is lower than that of a dwarf star of the same spectral type. The spectrum of the nova near maximum can be classed as F5 with super-giant characteristics. The effective temperature at maximum is therefore probably somewhat lower than that of a normal giant star of F5 type. The error in the adopted effective temperature near maximum should therefore be small.

The magnitude estimates made at La Plata * have been used as a basis for the determination of the rate of expansion of the nova between 1925 May 27 and June 8. The La Plata magnitudes are based upon revised values of the Harvard magnitudes for the comparison stars and form a homogeneous series. They show a practically uniform rise in brightness. The rate of expansion is assumed to have been uniform during this period ; the constant displacement shown by the spectrum lines is justification for this assumption. The effective temperature of the nova before the outburst is not known ; it will be assumed, in the first place, that there is no change of effective temperature between the outburst and maximum, but a correction will subsequently be

* *Ast. Nach.*, **226**, 203, 1926 (No. 5413).

introduced to give the rate of expansion for any assumed initial effective temperature.

The observed photographic magnitude before the outbreak was $12^m{\cdot}75$. The corresponding visual magnitude will be taken as $12^m{\cdot}28$.

With these assumptions the rate of increase in radius per day between the time of discovery and the first observed maximum was determined by a least squares solution and found to be

$$\Delta R = 5{\cdot}77 R_0.$$

If now the assumption of no change in effective temperature is abandoned, the corresponding rate of expansion will become

$$\Delta R = 5{\cdot}77 R_0 T_0^2/T^2,$$

but T_0 remains unknown. The temperature, T, during the interval between discovery and maximum brightness is assumed to be constant.

The deduced value will be affected by any errors in the adopted photographic magnitude before outbreak, and in colour-index. If Δm_0 is the correction to the adopted visual magnitude before outbreak, the correction to the derived rate of expansion to allow for the Δm_0 correction is easily obtained. For we have

$$m_0 - m = 5 \log RT^2/R_0T_0^2.$$

If we write

$$m_0 + \Delta m_0 - m = 5 \log RT^2(1 + x)/R_0T_0^2$$

then

$$\log (1 + x) = \Delta m_0/5,$$

and the corrected value of ΔR is thus obtained by multiplying the previously derived value by $(1 + x)$.

We may therefore adopt as the rate of expansion prior to the first maximum

$$\Delta R = 5{\cdot}77(1 + x)\frac{T_0^2}{T^2}R_0.$$

We now turn to the information provided by the spectroscopic observations. These indicate that between May 25 and June 5 there was a uniform expansion of the photosphere of the star with a velocity of approach of 65 km./sec. It has been shown that the radial velocity of the star is a velocity of recession of 12 km./sec., and the true Doppler displacement is therefore one corresponding to a velocity of 77 km./sec. The measured Doppler displacement is an integrated value for the whole disc, and must be increased to obtain the true rate of expansion. The correcting factor depends upon the law of darkening. If this is of the form $(a + b \cos \theta)$, the factor is $(6a + 4b)/(4a + 3b)$.* If darkening across the disc is neglected, the factor is 3/2. If the law of darkening found for the Sun $(2 + 3 \cos \theta)$ is used, the factor is 24/17. In the case of a purely normal outrush of light the correcting factor would be 4/3. Eddington has remarked, with reference to the pulsation of the Cepheid

* *Vide* Carroll, *M.N.R.A.S.*, **88**, 550, 1928.

variables, that the pulsation may cause the light at maximum to rush out more in a normal direction than in a static star,* and the same remark probably applies to an expanding nova. The three multiplying factors, corresponding to the three cases considered, are respectively 24/16, 24/17, and 24/18. Taking the two extreme cases, (*a*) uniform brightness, (*b*) complete darkening at limb, the deduced rates of increase per day in the radius are found to be

$$(a)\ \Delta R = 14{\cdot}4R_\odot$$
$$(b)\ \Delta R = 12{\cdot}8R_\odot$$

where $R_\odot$ denotes the Sun's radius. We will adopt the mean value of $13{\cdot}6R_\odot$. Combining this with the expression for ΔR in terms of the initial radius R_0, found above, there results

$$R_0 = \frac{2{\cdot}36}{(1+x)}\frac{T^2}{T_0{}^2}R_\odot.$$

The absolute visual magnitude of the star in the pre-nova state is given with close approximation by Russell's formula : †

$$M = \frac{29500}{T_0} - 5\log\frac{R_0}{R_\odot} - 0{\cdot}08.$$

The apparent visual magnitude being $12{\cdot}28 + \Delta m_0$, we have

$$12{\cdot}28 + \Delta m_0 + 5 + 5\log\pi = \frac{29500}{T_0} + 10\log\frac{T_0}{T} + 5\log(1+x) - 1{\cdot}95$$

or

$$5\log\pi = \frac{29500}{T_0} + 10\log\frac{T_0}{T} - 19{\cdot}23.$$

It will be noticed that errors in the adopted initial photographic magnitude and colour-index do not affect the derived value of the parallax. This is because with change in these initial quantities the derived value of R_0 is altered and the consequent change in $-5\log R_0/R_\odot$ exactly counteracts the concurrent change in M.

Adopting $T = 5700°$ and trying various values for T_0, we have the following figures :—

T_0.	$\frac{29500}{T_0}$.	$10\log\frac{T_0}{T}$.	$5\log\pi+15$.	$\log\pi+3$.	π.
7000°	4·21	0·89	0·87	0·174	0″·00149
6750	4·37	0·73	0·87	0·174	·00149
6500	4·54	0·57	0·88	0·176	·00150
6250	4·72	0·40	0·89	0·178	·00151
6000	4·92	0·22	0·91	0·182	·00152

The derived value of π is thus practically independent of the initial (unknown) value of T_0, and we can adopt as the parallax of Nova Pictoris the value

$$\pi = 0''{\cdot}0015.$$

* *Internal Constitution of the Stars*, p. 185.

† Russell, Dugan, and Stewart, *Astronomy*, p. 732 (Ginn & Co., 1927).

The initial radius and rate of expansion (expressed in terms of the initial radius) are dependent upon the value of T_0. In general nothing is known as to the spectral type of a nova before its outbreak. In the case of Nova Aquilæ No. 3 (1918) the spectrum before outbreak was of type A and at maximum was *c*A2. It appears probable that, in general, in a nova outburst there is not a great change in spectral type, but mainly a change from dwarf to giant characteristics for the same type. The most reasonable value to adopt for the initial effective temperature of Nova Pictoris would thus appear to be the normal value for a dwarf F5 spectrum, say 6500°. With this initial temperature it follows that

$$R_0 = 1{\cdot}82R_\odot.$$

Also that, at maximum, when the bolometric apparent magnitude was $1^m{\cdot}23$,*

$$\begin{aligned} R &= 211R_0 \\ &= 384R_\odot. \end{aligned}$$

The corresponding figures for various other values of T_0 are as follows :—

T_0	$R_0/R_\odot$.	R/R_0.	$R/R_\odot$.
7000°	1·51	253	382
6750	1·66	231	383
6500	1·82	211	384
6250	2·01	191	384
6000	2·21	173	382

The figures in the last column are constant, as they should be, for the rate of expansion given by the spectroscopic data is independent of T_0, and the time required for the star to expand to its radius at maximum brightness must also be independent of T_0.

As a general check on the above figures comparison can be made with the bolometric magnitude of the Sun. Assuming that the Sun's bolometric magnitude is 4·83 and that the effective temperature of the Sun is equal to that of the nova at maximum, 5700°, it follows that the radius of the nova at maximum was 363 times the radius of the Sun, in fair agreement with the above figure.

The parallax, 0″·0015, which has been derived above, corresponds to a visual absolute magnitude at maximum brightness of − 7·9, comparable with the absolute magnitudes observed for previous novæ. Davidovich estimated the absolute magnitude at maximum as − 5·0, which corresponds to a parallax of 0″·006. It is not possible to reconcile a parallax of this magnitude with the hypothesis of an expansion of the star between the time of discovery and the first subsequent maximum.

* The La Plata observations give maximum June 10·0, with visual magnitude $0^m{\cdot}96$. The estimate of $1^m{\cdot}23$ is based upon all available observations. The radius at maximum depends upon the value used for the magnitude at maximum.

The visual absolute magnitude of the nova before the outbreak was 3·15, a normal value for a dwarf F-type star.

It remains to examine whether a parallax of 0″·0015 is in reasonable agreement with the observations of the diameter of the nebulosity surrounding the nova observed by van den Bos and Finsen at the Union Observatory,* Johannesburg. The visual observations of the nebulosity are not easy on account of its faintness and the disturbing effect of the companion stars. Van den Bos states that the nebulous disc is irregular and of varying surface brightness, with a distinct drop in brightness between the nuclei. It is roughly oval in shape, with the maximum diameter in the approximate direction 80°–260°, and in the perpendicular direction a diameter equal to or rather less than two-thirds of the maximum diameter. Between 1928 and 1930 it faded considerably.

Van den Bos states that in 1928 the nebulosity seemed definitely to extend beyond B and C ; in 1930 B seemed still involved, but C on edge, or even just outside. The mean distances of B and C from the central star A were approximately as follows :—

	AB.	AC.	No. of Nights.
	″	″	
1928·33	0·4 ±	0·4 ±	4
1929·07	0·63	0·68	4
1930·11	0·80	0·82	2

It may therefore be concluded that the radius in the direction BC was of the order of, say, 0″·45 in 1928 and of 0″·80 in 1930. The direction BC is approximately the direction of maximum diameter.

Assuming a parallax of 0″·0015 and a mean velocity of expansion of 430 km./sec. (as derived from the spectroscopic observations), the diameter of the nebulous shell should expand at a rate of 0″·27 per year. In 1928, three years after the outburst, the radius of the shell would be 0″·41, and in 1930, five years after the outburst, it would be 0″·68. There is fair agreement with the Johannesburg observations when the vagueness of the nebulosity and the difficulty of the observations are taken into consideration. A parallax of 0″·0018 would give a somewhat better representation. The observations are difficult to explain fully, but they do not appear to be inconsistent with a parallax of the order of 0″·0015. If, on the other hand, the parallax was 0″·006, the diameter of the nebulous shell would be observed to expand at a rate of 1″·08 per year, and the radius of the shell in 1930 would have been nearly 3″. The Johannesburg observations appear definitely to disprove this.

Broad emission bands first appear in the spectrum at or just after the first observed maximum. It would seem, therefore, that gaseous shells were thrown off about that time. The radius of the nova being, as we have seen, about 380 times the radius of the Sun, the surface gravity would be low and the velocity of escape under gravity alone would be correspondingly low. There does not appear to be any

* Mr. H. E. Wood, Union Astronomer, kindly supplied me with a manuscript copy of the observations made by Dr. van den Bos and Mr. Finsen.

difficulty, on Milne's theory, in supposing that at this stage shells of gas were thrown off with the velocities actually observed. The two principal absorptions present in the spectra of the nova at this stage are Absorption II., displacement about 300 km./sec., and Absorption III., displacement about 120 km./sec.*

Support for this view can be obtained from the spectroscopic observations. Immediately after a shell is thrown off, the diameter of the shell is not much greater than that of the star itself. The emission bands are produced by the light from that portion of the shell which lies outside the boundary of the star, as seen by the observer. The motion of this portion of the shell is mainly perpendicular to the line of sight, and therefore the emission bands will not at first have the full width which corresponds to the velocity of the expanding shell. As the shell expands further, an increasing proportion of its surface contributes to the emission and the breadths of the emission bands should gradually increase. The emission bands, when they first appear, are, however, weak and much broken into by absorption lines, so that it is not possible to measure their edges with any great accuracy.

Another method of approach is available in the absorption lines, which provide complementary information. The position of an absorption line depends upon the average velocity in the line of sight of the portion of the shell which is seen in projection on the disc of the star. As the shell expands this average velocity increases, and the centres of the absorption lines should move to the violet. This was actually observed in the case of Nova Pictoris after the first maximum.

The numerical investigation depends upon the law of darkening assumed for the shell. The best assumption is probably that of the normal emission of light. If V is the velocity of expansion, it can be easily shown that the observed displacement of the absorption lines will be

$$\tfrac{3}{4}V\frac{1-\cos^4\theta}{1-\cos^3\theta},$$

where $\sin\theta$ = radius of star/radius of shell.

If, on the other hand, the assumption of uniform brightness across the disc had been made, the corresponding velocity would have been

$$\tfrac{2}{3}V\frac{1-\cos^3\theta}{1-\cos^2\theta}.$$

It is a matter of indifference which law of darkening is adopted, as by a suitable adjustment of the data a satisfactory representation of the observed velocities can be obtained on either law. For the purpose of the illustrative computation the assumption has been made of complete darkening at the limb. The available data do not suffice to provide a unique representation of the observed velocities, as a tolerably good fit can be obtained with considerable variation in the assumed initial conditions. In particular, the observed velocity displacements do

* A full discussion of the spectroscopic material will appear shortly in the *Cape Annals.*

not provide any definite information as to the change of radius of the star. When, however, the initial radius of the star and its rate of change have been fixed, the radii of the expanding shells can be determined without great uncertainty.

Between June 9 and 28 the brightness of the star decreased by $2^m{\cdot}23$. This decrease must have been accompanied by a contraction of the star; the extent of the contraction required to account for the magnitude change depends upon the change in the effective temperature which accompanied it. This change was probably small, and, in the lack of any data, the assumption has been made that the temperature remained unaltered. The entire reduction in brightness was then due to the decrease in the surface area of the star. The amount of contraction in the radius of the star per day on this assumption is 12·5 times the Sun's radius. After various trials the finally adopted initial data are as follows :—

Radius of star on June 9	$370{\cdot}5 \times R_\odot$
,, inner shell on June 9 . . .	447·8
,, outer shell ,, . . .	592·0
Decrease in radius of star per day . .	12·50
Increase in radius of inner shell per day .	20·44
,, ,, outer ,, ,, .	55·20
Limiting velocity of inner shell . .	122·8 km./sec.
,, ,, outer ,, . . .	332·0 ,,

The velocities of the two shells correspond to the adopted rates of expansion.

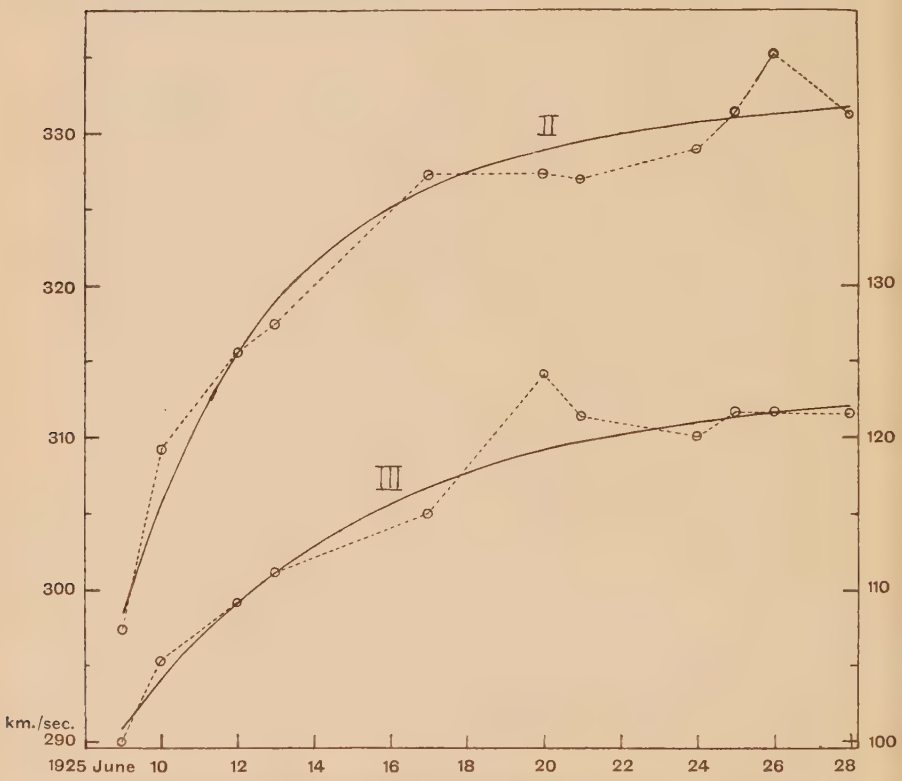
In the following table are given the computed radii of the star and the two shells for the period June 9–28 between first maximum and first minimum, and the resulting velocity displacements of the absorption lines produced by the absorption by the two shells respectively of light emitted by the star.

Computed Velocity Displacements of Absorptions II. and III.

1925.	Radius of Star.	Radius of Inner Shell.	Radius of Outer Shell.	Computed Velocity Absorption III. km./sec.	Computed Velocity Absorption II. km./sec.
June 9	370·5	447·8	592·0	100·8	298·5
10	358·8	468·2	647·2	104·1	305·9
12	333·0	509·1	758·6	109·2	315·8
14	308·0	550·0	870·0	112·9	321·5
16	283·0	590·9	981·4	115·6	325·1
18	258·0	631·8	1092·8	117·6	327·4
20	233·0	672·6	1204·2	119·1	328·9
22	208·0	713·5	1314·6	120·1	330·0
24	183·0	754·4	1425·0	121·0	330·7
26	158·0	795·3	1535·4	121·6	331·2
28	133·0	836·2	1645·8	122·0	331·6

The comparison between the computed and observed velocities is given in the following table and is illustrated in the figure. In the

figure the velocity scale on the left applies to Absorption II. and that on the right to Absorption III.



Increase in velocity displacements of Absorptions II. and III., 1925 June 9–28.

Comparison between Computed and Observed Velocities.

1925.	Absorption III.			Absorption II.		
	Observed Velocity. km./sec.	Computed Velocity. km./sec.	O−C. km./sec.	Observed Velocity. km./sec.	Computed Velocity. km./sec.	O−C. km./sec.
June 9	100·0	100·8	−0·8	297·3	298·5	−1·2
10	105·4	104·1	+1·3	309·1	305·9	+3·2
12	109·1	109·2	−0·1	315·7	315·8	−0·1
13	111·2	111·2	0·0	317·5	319·0	−1·5
17	115·0	116·2	−1·2	327·2	326·4	+0·8
20	124·1	119·1	+5·0	327·3	328·9	−1·6
21	121·4	119·6	+1·8	326·9	329·5	−2·6
24	120·0	121·0	−1·0	328·9	330·7	−1·8
25	121·6	121·3	+0·3	331·4	331·0	+0·4
26	121·6	121·6	0·0	335·2	331·2	+4·0
27	121·5	121·8	−0·3	331·1	331·4	−0·3

The close general agreement between the observed and computed velocities lends strong support to the view that up to the time of the first observed maximum the nova was expanding, that about the time of maximum at least two gaseous shells were thrown off, and that the observed Absorptions II. and III. originated in the absorption by these shells of light emitted by the nova.

Summary.

1. A trigonometrical determination of the parallax of Nova Pictoris gave the value $-0''{\cdot}009 \pm 0''{\cdot}007$ for the absolute parallax.

2. On the assumption that up to the time of the first observed maximum the nova was expanding, a parallax of $0''{\cdot}0015$ has been derived by correlating the increase in brightness with the rate of expansion given by the spectroscopic data. The corresponding absolute visual magnitude at maximum is $-7{\cdot}9$.

3. This value of the parallax is independent of the unknown effective temperature of the nova before the outburst. It is not inconsistent with the observations of nebulosity surrounding the nova made at the Union Observatory, Johannesburg.

4. The radius of the nova before the outburst was of the order of $1{\cdot}8$ times the Sun's radius. At maximum the radius was about 384 times the Sun's radius.

5. Gaseous shells appear to have been emitted near the time of maximum. The progressive increase in the observed velocity displacements of the two principal absorptions present in the spectrum after first maximum can be theoretically accounted for by the spreading outwards of these shells combined with the contraction of the nova.

The Slow Contraction or Expansion of a Fluid Sphere. II. Stability.

By L. H. Thomas, M.A., Ph.D.

1. *Introduction.*—In my first paper (*M.N.*, **91**, 122, 1930), which will be referred to as I., the equilibrium of slow radial motion of a fluid sphere with conduction of heat but without internal generation of energy was discussed. It was noted there (I., 9) that internal generation of energy would complicate the analysis; the stability conditions, however, are but little altered, and the more general case will be considered below.

The equations of motion (I., 5.1, 5.2, 5.3) are :—

$$\ddot{r} = 4\pi r^2 \frac{d}{dm}\frac{\partial U}{\partial \sigma} - \frac{\gamma m}{r^2} \quad . \quad . \quad . \quad . \quad (1.1)$$

$$\frac{\partial U}{\partial s}\dot{s} = 16\pi^2 \frac{dm}{d}\left\{\frac{cr^4}{3\kappa}\frac{d}{dm}a\left(\frac{\partial U}{\partial s}\right)^4\right\} + g \quad . \quad . \quad (1.2)$$

$$\sigma = 4\pi r^2 \frac{dr}{dm} \quad . \quad . \quad . \quad . \quad . \quad . \quad (1.3)$$

These hold for $0 \leqslant m \leqslant M$, with boundary conditions $r = 0$ for $m = 0$, and $\partial U/\partial\sigma = 0$ for $m = M$.

Here r is the radius of the sphere occupied by mass m. σ is the volume, s the entropy, and $U(s, \sigma)$ the internal energy of the matter and radiation per unit mass of the matter. κ is the opacity of the matter, γ the constant of gravitation, and c the velocity of light. a is Stefan's constant.

g, which represents the rate of generation of energy per unit mass at the point considered, was left out in I. (5.2) and is supposed here to depend on s, σ, and perhaps also on m and t explicitly. We shall suppose that if g depends explicitly on t, $\partial g/\partial t$ is very small compared with g, being at most of order g^2/U.

We have a conservative dynamical system with a Lagrange function

$$L = \int_0^M \left\{\tfrac{1}{2}\dot{r}^2 - U(s, \sigma) + \gamma\frac{m}{r}\right\}dm \quad . \quad . \quad (1.4)$$

where

$$\sigma = 4\pi r^2\frac{dr}{dm},$$

the potential energy of which involves a parameter s which is very slowly changing. At any time s is a definite function of m, but its rate of change depends on the state of the system at that time and perhaps also on t explicitly. We wish to investigate the stability of slow change near an equilibrium solution of (1.1, 1.3).

2. *Systems with One Degree of Freedom.*—The equations for the analogous problem with one degree of freedom instead of the continuous set represented by $r(m)$ are

$$\ddot{r} + \frac{\partial}{\partial r}V(r, s) = 0 \quad . \quad . \quad . \quad . \quad (2.1)$$

$$\dot{s} = \epsilon f(r, s, t) \quad . \quad . \quad . \quad (2.2)$$

where $V(r, s)$ is potential energy per unit mass and ϵ is a very small number. We assume that $\partial f/\partial t$ is small compared with f. We require the conditions that if $\dot{r}$ is small initially and r and s are nearly equal to r_0 and s_0, where r_0, s_0 satisfy

$$\frac{\partial}{\partial r_0}V(r_0, s_0) = 0 \quad . \quad . \quad . \quad . \quad . \quad (2.3)$$

$$\dot{s}_0 = \epsilon f(r_0, s_0, t) \quad . \quad . \quad . \quad (2.4)$$

it shall follow that r and s remain nearly equal to r_0 and s_0, *i.e.* that they differ by less than η after time of order $1/\epsilon$, where $\eta \to 0$ as $\epsilon \to 0$.

First, it is necessary that the motion given by (2.1) for constant s should be stable near $r = r_0$. This condition of "dynamical stability" is simply that $V(r, s)$ should be a true minimum for $r = r_0$, *i.e.* that

$$\frac{\partial^2}{\partial r_0^2}V(r_0, s) > 0 \quad . \quad . \quad . \quad . \quad (2.5)$$

That this condition is sufficient follows from the existence of the energy integral of (2.1) for constant s. To show that it is necessary would require the setting up, if it fails, of a solution with $\dot{r}$ and $r - r_0$ initially small and later not small. If the inequality is reversed, such a solution can be found.

If the condition of dynamical stability holds, the finite small motion near $r = r_0$ is periodic with, for infinitesimal motions, period $2\pi\sqrt{\left(1\Big/\frac{\partial^2 V}{\partial r_0^2}\right)}$. If, further, the rate of change of s does not depend on the state of the system, *i.e.* if $f(r, s, t)$ does not depend on r, so that s is a function of t given by

$$\dot{s} = \epsilon f(s, t),$$

then Ehrenfest's theorem * on the quasi-static (adiabatic) invariant for a simply periodic motion tells us that the change in

$$\int_0^T \tfrac{1}{2}\dot{r}^2 dt$$

integrated over a period of the motion for s constant, which therefore depends on the energy and on s only, is infinitesimal for a definite change in s if $\dot{s} \to 0$ and $\ddot{s}/\dot{s} \to 0$: provided only that the period T remains bounded. The way in which the energy depends on s and therefore on t is thus obtained, and the condition that the motion decays is that T increases.

The above condition holds for finite motions: for infinitesimal motions the condition is that $\partial^2 V/\partial r_0^2$ diminishes, or

$$\epsilon f \frac{\partial^3 V}{\partial r_0^2 \partial s} < 0 \qquad . \quad . \quad . \quad . \quad (2.6)$$

which can be obtained directly by considering

$$\frac{d}{dt}\{\tfrac{1}{2}\dot{r}^2 + (V - V_0)\}.$$

This condition is necessary and sufficient.

When the rate of change of s depends on the state of the system, two complications are introduced: $\ddot{s}/\dot{s}$ may not be small and s may depart from the solution of (2.3, 2.4). We now introduce a new value of r, viz. r_1, given by

$$\frac{\partial}{\partial r_1} V(r_1, s) = 0 \qquad . \quad . \quad . \quad . \quad (2.7)$$

Then

$$\frac{d}{dt}(\tfrac{1}{2}\dot{r}^2 + V - V_1) = \dot{s}\frac{\partial}{\partial s}(V - V_1)$$

$$\simeq \dot{s}\left\{\frac{\partial^2 V_1}{\partial r_1 \partial s}(r - r_1) + \frac{1}{2}\frac{\partial^3 V_1}{\partial r_1^2 \partial s}(r - r_1)^2\right\},$$

* This theorem, now generally known as Ehrenfest's theorem, was originally due to Boltzmann. See Ehrenfest, *Proc. Amsterdam Acad.*, **16**, 591, 1914.

keeping in no terms of order higher than $\epsilon(r - r_1)^2$. If, as above, $\dot{s}$ does not depend on r, the mean value of $\dot{s}(r - r_1)$ is zero and only the second term is important; in general

$$\dot{s} = \epsilon f(r, s, t) \backsimeq \epsilon \left[f(r_1, s, t) + \frac{\partial f}{\partial r}(r - r_1)\right]$$

and condition (2.6) is replaced by

$$\epsilon\left[f\frac{\partial^3 V}{\partial r^2 \partial s} + 2\frac{\partial f}{\partial r}\frac{\partial^2 V}{\partial r \partial s}\right] < 0 \quad . \quad . \quad . \quad (2.8)$$

If this is true, $(r - r_1)$ decreases exponentially.

Further

$$\frac{d}{dt}(s - s_0) = \epsilon[f(s, r, t) - f(s_0, r_0, t)]$$

$$\backsimeq \epsilon(s - s_0)\left[\frac{\partial f}{\partial s} - \frac{\dfrac{\partial^2 V}{\partial r \partial s}}{\dfrac{\partial^2 V}{\partial r^2}}\frac{\partial f}{\partial r}\right] + \epsilon(r - r_1)\frac{\partial f}{\partial r}.$$

The term in $(r - r_1)$ is of higher order in the mean, and the condition that $(s - s_0)$ diminishes is

$$\epsilon\left[\frac{\partial f}{\partial s}\frac{\partial^2 V}{\partial r^2} - \frac{\partial f}{\partial r}\frac{\partial^2 V}{\partial r \partial s}\right] < 0 \quad . \quad . \quad . \quad (2.9)$$

If $\partial f/\partial t$ is not small there are further terms, the mean of $(r - r_1)\partial f/\partial r$ not necessarily being negligible.

The inequalities (2.5, 2.8, 2.9) are thus sufficient for r and s to remain nearly equal to r_0 and s_0, while if any one of these inequalities is reversed, r and s do not remain nearly equal to r_0 and s_0. Conditions (2.5), (2.8) are the conditions of dynamical stability and thermodynamical stability, and (2.9) is the further condition of stability mentioned in I.

3. *Systems with more than One Degree of Freedom.*—For systems with several degrees of freedom, the condition of "dynamical stability" is still that V should be a true minimum. The theory of quasi-static invariants has been extended only to those systems for which the motion is multiply periodic.* The state of affairs is then that solutions fall into groups according to their initial phase relations; for the principal group there is an invariant comparable to that of (2) for each degree of freedom. There are, however, also sets of solutions asymptotic to solutions with a smaller number of periods, for which the values of the invariants corresponding to the lost periods and phases are given by definite equations in terms of the other co-ordinates and the parameters. If there are more than two degrees of freedom, there will generally be sets of such degenerate solutions for all changes of the parameters.†

* Burgers, *Proc. Amsterdam Acad.*, **20**, 163, 1917.

† Dirac, *Proc. Roy. Soc.*, A, **107**, 725, 1925; L. H. Thomas, *Proc. Camb. Phil. Soc.*, **22**, 895, 1925.

It is unlikely that the state of affairs will be any less complicated when the rates of variation of the parameters depend on the state of the system at the time. For sufficiently small oscillations about equilibrium, however, it can be shown by a similar argument to that in (2) that the changes of the normal oscillations do not interfere with one another. There will therefore be a condition of type (2.8) for each normal mode. If these conditions were satisfied, it would of course not follow that *finite* pulsations would diminish.

Finally, the rates of variation of the parameters themselves from their expected values will be given by equations of the form

$$\frac{d}{dt}\delta s_1 = A_{11}\delta s_1 + A_{12}\delta s_2,$$

$$\frac{d}{dt}\delta s_2 = A_{21}\delta s_1 + A_{22}\delta s_2,$$

and there will be conditions of type (2.9) that the solutions of these equations remain small.

The conditions of the second type considered above are not easy to apply, but a condition which is sufficient, though not necessary, for thermodynamical equilibrium will be obtained below. The conditions of the third type form, in general, a set of linear integro-differential equations that can only be attacked directly.

4. *The Conditions of Dynamical Stability.*—The condition that a configuration $r = r(m)$ is a configuration of stable equilibrium for motions given by (1.1, 1.3) is that

$$\int_0^M \left\{ -\frac{\gamma m}{r} + U(\sigma) \right\} dm$$

is a minimum, where $\sigma = 4\pi r^2 dr/dm$. The conditions for this are * :—

(1) The Euler condition,

$$-\frac{\gamma m}{r^2} + 4\pi r^2 \frac{d}{dm}\frac{\partial U}{\partial \sigma} = 0 \qquad \text{for } 0 \leqslant m \leqslant M,$$

$$r^2 \frac{\partial U}{\partial \sigma} = 0 \qquad \text{for } m = 0 \quad \text{and} \quad m = M \qquad (4.1)$$

the characteristic equation and the boundary condition.

(2) The Legendre condition,

$$\frac{\partial^2 U}{\partial \sigma^2} > 0 \qquad \text{for } 0 \leqslant m \leqslant M.$$

(3) The Weierstrass condition,

$$U(\lambda) - U(\sigma) - (\lambda - \sigma)\frac{\partial U(\sigma)}{\partial \sigma} > 0 \qquad \text{for } 0 \leqslant m \leqslant M \text{ and all } \lambda.$$

* Forsyth, *Calculus of Variations*, p. 385, 1927.

(4) The Jacobi condition, starting at $m = 0$, a consecutive solution of the characteristic equation satisfying the boundary condition at $m = 0$, must not lead to the same value of r for a given value of m as the solution satisfying both boundary conditions for any $m \leqslant M$.

Conditions (1) are the conditions of equilibrium. Conditions (2) and (3) are the conditions that the matter at any point is dynamically stable. Condition (4) is the important condition of dynamical stability for the whole solution and is in a form that is usually easy to apply, since it is usually necessary to carry out a number of quadratures with various initial conditions to obtain that solution which satisfies both boundary conditions.

For polytropic spheres for which $U = a\sigma^{-1/k}$, the consecutive solutions are homologous with

$$\delta r = \beta \frac{1-n}{3-n} m^{-\frac{2}{3-n}} \delta m$$

and the condition gives $n < 3$, the usual result.*

The solutions satisfying the second boundary condition could be used just as well. Both sets will be required below.

5. *The Conditions of Thermodynamical Stability.*—The conditions which, following Jeans,† are here called those of thermodynamical stability are that the changes in the values of s owing to generation and transfer of energy do not cause the energy of any normal oscillation about the state of dynamical equilibrium to increase.

The rate of change of the dynamical energy is

$$\frac{d}{dt}\int_0^M \left(\tfrac{1}{2}\dot{r}^2 + U - \frac{\gamma m}{r} - U_1 + \frac{\gamma m}{r_1}\right) dm = \int_0^M \left[\frac{\partial U}{\partial s} - \frac{\partial U_1}{\partial s}\right] \dot{s}\, dm$$

$$\fallingdotseq \int_0^M \left[(\sigma - \sigma_1)\frac{\partial^2 U}{\partial s \partial \sigma} + \tfrac{1}{2}(\sigma - \sigma_1)^2 \frac{\partial^3 U}{\partial s \partial \sigma^2}\right] \dot{s}\, dm$$

where

$$\dot{s}\frac{\partial U}{\partial s} = 16\pi^2 \frac{d}{dm}\left\{\frac{cr^4}{3\kappa}\frac{d}{dm} a\left(\frac{\partial U}{\partial s}\right)^4\right\} + g \quad . \quad . \quad (1.2)$$

The terms of lowest order which do not vanish in the mean are then, replacing $\sigma - \sigma_1$ by $\delta\sigma$, and $r - r_1$ by δr, and not distinguishing r and r_1, σ and σ_1 in their coefficients,

$$\int_0^M \left\{\tfrac{1}{2}(\delta\sigma)^2 \frac{\partial^3 U}{\partial s \partial \sigma^2}\dot{s} + \delta\sigma \frac{\partial^2 U}{\partial s \partial \sigma}\delta\dot{s}\right\} dm$$

where $\dot{s}$ is given by 1·2.

If $v = \frac{4}{3}\pi r^3$, so that $\sigma = dv/dm$, this takes the form

$$\int_0^M [A\delta v \delta v' + B\delta v'^2 + C\delta v' \delta v'' + D\delta v' \delta v''']dm$$

where primes represent differentiations with respect to m, and where, in particular,

* Eddington, *The Internal Constitution of the Stars*, p. 142, 1926.

† *Astronomy and Cosmogony*, p. 140, 1928.

$$\mathrm{D} = \frac{16\pi^2 ac}{\kappa} r^4 \left(\frac{\partial \mathrm{U}}{\partial s}\right)^2 \left(\frac{\partial^2 \mathrm{U}}{\partial s \partial \sigma}\right)^2$$

and is essentially positive.

The last term can be integrated by parts, leaving terms

$$-\mathrm{D}\delta v''^2 - \frac{d\mathrm{D}}{dm}\delta v'\delta v''$$

and the expression is then exactly similar to the second variation of an integral of the form $\int_0^{\mathrm{M}} \mathrm{F}(m, v, v', v'')dm$, the coefficients of the variations being known functions of m. Since there are no necessary relations between the coefficients of such variation as functions of m, the theory of these variations * can be used to modify the form of the expression.

The method depends on solving the linear equation

$$\frac{\partial \Omega}{\partial \eta} - \frac{d}{dm}\frac{\partial \Omega}{\partial \eta'} + \frac{d^2}{dm^2}\frac{\partial \Omega}{\partial \eta''} = 0 \quad . \quad . \quad . \quad (5.1)$$

where in this case

$$\Omega = \mathrm{A}\eta\eta' + \mathrm{B}\eta'^2 + \left(\mathrm{C} - \frac{d\mathrm{D}}{dm}\right)\eta'\eta'' - \mathrm{D}\eta''^2.$$

Two solutions, α, β, of this equation satisfying a certain condition being obtained, as is always possible for essentially positive D,

$$\int_0^{\mathrm{M}} [\mathrm{A}\delta v\delta v' + \mathrm{B}\delta v'^2 + \mathrm{C}\delta v'\delta v'' + \mathrm{D}\delta v'\delta v''']dm$$
$$= -\int_0^{\mathrm{M}} \frac{\mathrm{D}}{(\alpha\beta' - \alpha'\beta)^2} \begin{vmatrix} \delta v'' & \delta v' & \delta v \\ \alpha'' & \alpha' & \alpha \\ \beta'' & \beta' & \beta \end{vmatrix}^2 dm$$
$$+ \Big|_0^{\mathrm{M}} \left| \theta\delta v'^2 + \phi\delta v\delta v' + \psi\delta v^2 + \mathrm{D}\delta v\delta v'' \right|$$

where θ, ϕ, ψ depend on α and β.

At the boundaries it is requisite for a normal oscillation that $\delta(r^2\partial \mathrm{U}/\partial\sigma)$ should vanish as well as $r^2\partial \mathrm{U}/\partial\sigma$, and this suffices to make the terms at the limits in the above expression vanish.

Since D is essentially positive, the dynamical energy must diminish except for such δv, $\delta v'$, $\delta v''$ that

$$\begin{vmatrix} \delta v'' & \delta v' & \delta v \\ \alpha'' & \alpha' & \alpha \\ \beta'' & \beta' & \beta \end{vmatrix}$$

vanishes, and, just as in the calculus of variations, we see that no such variations exist if the solution of (5.1) for which $\eta = 0, \eta' = 0, \eta'' = 0$, $\eta''' = 1$ for $m = 0$, does not again vanish in the range $0 \leqslant m \leqslant \mathrm{M}$.

* Forsyth, *loc. cit.*, p. 116.

The above condition is sufficient for thermodynamical stability; it does not require solving the equations for small oscillations. If the condition is not satisfied it is probable that some oscillation will be unstable.

The terms involving g are of the form

$$\int_0^M \frac{(\delta\sigma)^2}{\left(\frac{\partial U}{\partial s}\right)^2}\left[g\left\{\frac{1}{2}\frac{\delta^3 U}{\partial s\partial\sigma^2}\frac{\partial U}{\partial s}-\frac{\partial^2 U}{\partial s\partial\sigma}\right\}+\frac{\partial g}{\partial\sigma}\frac{\partial U}{\partial s}\frac{\partial^2 U}{\partial s\partial\sigma}\right]dm$$

$$=\tfrac{1}{2}\int_0^M \frac{\frac{\partial U}{\partial s}}{g}(\delta\sigma)^2\frac{d}{d\sigma}\left[\frac{g^2\frac{\partial^2 U}{\partial s\partial\sigma}}{\left(\frac{\partial U}{\partial s}\right)^2}\right]dm.$$

The coefficient of $(\delta\sigma)^2$ depends only on the nature and state of the matter at the point considered. If it is positive there is possible instability; if it is positive and sufficiently large there will certainly be instability unless there is sufficient compensation by a high rate of cooling. An exact discrimination, of course, requires the whole process outlined to be carried through, and there might be instability even with $g = 0$.

For a polytropic sphere with $U = a(s)\sigma^{-1/k}$, since da/ds must be positive, generation of energy independent of physical conditions would be expected to set up instability unless $k < 1$. For $k = 3$, $\partial g/\partial\sigma > g/3$ if there is to be no tendency to instability.

6. *Stability Conditions of the Third Type.*—If the equilibrium is dynamically and thermodynamically stable there are still conditions of type (2.9) to be satisfied. We first find $r_1 - r_0$ in terms of $s - s_0$. We require to know how the solution of

$$-\frac{\gamma m}{r^2}+4\pi r^2\frac{d}{dm}\frac{\partial U}{\partial\sigma}=0 \qquad 0\leqslant m\leqslant M$$

$$r^2\frac{\partial U}{\partial\sigma}=0 \qquad m=0 \text{ or } m=M \quad . \quad (4.1)$$

varies with s, and therefore use the linear subsidiary equation

$$\frac{2\gamma m}{r^3}\delta r+8\pi r\delta r\frac{d}{dm}\frac{\partial U}{\partial\sigma}+8\pi r^2\frac{d}{dm}\left\{\frac{\partial^2 U}{\partial\sigma^2}\delta\sigma\right\}=0 \quad . \quad (6.1)$$

Let ξ, η be the solutions of this equation satisfying the boundary condition

$$2\delta r\frac{\partial U}{\partial\sigma}+r\frac{\partial^2 U}{\partial\sigma^2}\delta\sigma=0 \quad . \quad . \quad . \quad (6.11)$$

at $m = 0$ and $m = M$ respectively, where

$$\delta\sigma=8\pi r\delta r\frac{dr}{dm}+4\pi r^2\frac{d}{dm}\delta r \quad . \quad . \quad . \quad (6.12)$$

If the solution of (4.1) has been obtained by quadrature, one of these solutions will usually be known and the other can then be obtained by integration. They satisfy, indeed,

$$4\pi r^2 \frac{\partial^2 U}{\partial \sigma^2}\left(\xi \frac{d\eta}{dm} - \eta \frac{d\xi}{dm}\right) = a \quad . \quad . \quad . \quad (6.2)$$

where a is a constant.

The change in the solution of (4.1) due to change δs in s is then

$$\delta r = \eta \int_0^M \frac{1}{a} \frac{\partial^2 U}{\partial \sigma \partial s} \delta s d\xi + \xi \int_m^M \frac{1}{a} \frac{\partial^2 U}{\partial \sigma \partial s} \delta s d\eta \quad . \quad . \quad (6.3)$$

The change in δs with time is

$$\frac{d}{dt}\delta s = \delta\left[\left\{16\pi^2 \frac{d}{dm}\frac{cr^4}{3\kappa}\frac{d}{dm}a\left(\frac{\partial U}{\partial s}\right)^4 + g\right\}\Big/\frac{\partial U}{\partial s}\right] \quad . \quad (6.4)$$

wherein δr is given by 6.3.

The right-hand side of this equation unfortunately contains δs under integral signs as in (6.3); that is, the rate of change of δs at any point depends on its values at all points and not only on its values at neighbouring points. The methods of the calculus of variations leading to simple results of the types obtained for dynamical and thermodynamical equilibrium can therefore not be applied. It is necessary to show directly that the values of λ which make the integro-differential equation

$$\lambda \delta s = \delta\left[\left\{16\pi^2 \frac{d}{dm}\frac{cr^4}{3\kappa}\frac{d}{dm}a\left(\frac{\partial U}{\partial s}\right)^4 + g\right\}\Big/\frac{\partial U}{\partial s}\right] = \delta \dot{s}$$

soluble are all negative. This would, for example, be true if we could find a function $\phi(m)$ such that

$$\int_0^M \phi \delta s \delta \dot{s} dm$$

is essentially negative, while if such an expression were large and positive it would be a strong indication that the motion was unstable.

7. *Summary.*—The stability of the slow symmetrical contraction or expansion of a fluid sphere with conduction of heat and internal generation of energy is considered. The conditions of stability fall into three groups.

(1) The conditions of dynamical stability; these are the well-known conditions that the dynamical energy should be a true minimum. It is pointed out that the calculation necessary to apply them has already been done if the equilibrium state has been obtained by quadrature.

(2) The conditions of thermodynamical stability; the necessary conditions, that the slow changes should diminish the energy of every normal dynamical oscillation, are difficult to apply. A sufficient condition which requires only the solution of a linear differential equation with definite initial conditions is obtained. If this condition is far from being satisfied, stability is improbable.

(3) The equations which determine, in case of dynamical and thermodynamical stability, small variations from the expected slow change are set up. It appears that they are of a rather general integro-differential type, and no simple criterion of stability emerges.

The results agree generally with those obtained by Jeans and Vogt * in particular cases by constraining the variations to be homologous, but apply also to other variations. The important point is the existence of conditions of the third type.

Pressures at the Base of the Chromosphere: a Critical Study of Milne's Theories. By Donald H. Menzel.

(*Communicated by Dr. R. G. Aitken, Assoc. R.A.S.*)

§ 1. *The General Problem of the Chromosphere.*—Kirchhoff's suggestion that the solar atmosphere can be represented by a series of successive layers has never been superseded, although our concept of the physical constitution of the various strata has altered markedly. Four divisions are generally recognized. There is the *photosphere*, no longer considered a liquid surface, which produces the continuous spectrum. Surmounting this is the *reversing layer*, which gives rise to the Fraunhofer absorption. Next above is the *chromosphere*, a name given by Lockyer to the layer that is characterized by a monochromatic emission spectrum, best observed at a total solar eclipse. Highest of all is the *corona*.

This elementary review is important for several reasons. First of all, I wish to emphasize the *original definitions* of the various words, particularly of the word *chromosphere*. In the second place, I wish to point out that the distinction between *reversing layer* and *chromosphere* has never been satisfactorily defined. This question was the chief bone of contention between earlier observers, Lockyer even maintaining that the locus of absorption was *above* rather than *below* the chromosphere. The observational grounds for his view, viz., the non-complementary character of the flash and Fraunhofer spectra, have been upheld, but the objections he advanced against the low-lying reversing layer have been entirely answered by the theories of ionization.

A few years ago Milne † introduced, rather arbitrarily it seems to me, new definitions for the reversing and chromospheric layers. Though, of course, there is no sharp line of demarcation between the two, Milne supposes the former to be in what he calls "*local thermodynamic equilibrium*" and the latter to be in "*monochromatic radiative equilibrium.*" According to Milne, "matter in the neighbourhood of a point P will be said to be in *local thermodynamic equilibrium* at temperature T when its emission of radiation is the same as if it were in strict thermodynamic equilibrium at temperature T, *i.e.* is the same as

* H. Vogt, *Veröffentlichungen der Universitäts-Sternworte zu Jena*, **2**, 1929.

† *M.N.*, **88**, 188, 1928.

if it were enclosed in a uniform temperature enclosure at temperature T. Then Kirchhoff's law holds strictly." On the other hand, "matter in the neighbourhood of a point P is said to be in *monochromatic radiative equilibrium* in respect of ν-radiation when it re-emits all the ν-radiation it absorbs," without change of frequency.

These definitions, when applied to the sun, require justification. The essential distinction between Milne's two types of equilibrium is that collisions are relatively active in the former and relatively rare in the latter, with respect to radiative processes. In other words, the effect of collisions is to produce a continuous spectrum. Since the quantum theory of collisions is in a rather unsatisfactory state, precise numerical values cannot be obtained. The present paper will be concerned with an analysis of Milne's theories * of chromospheric equilibrium and with the relation of these theories to the sun itself. We shall attempt to answer the following questions in particular: Are Milne's types of equilibrium possible physically? Do any of them occur upon the sun? Does any one of Milne's theoretical layers correspond to the observed chromosphere? What is the pressure at the base of the observed chromosphere?

Before proceeding to the explicit study of chromospheric conditions, we must pave the way by stating or developing certain fundamental physical formulæ.

§ 2. *The Atomic Absorption Coefficient.*—According to classical electromagnetic theory, a bound electron is capable of absorbing other frequencies, ν, than the one, ν_0, to which it is resonant. The atomic absorption coefficient, a_ν, so derived for ν-radiation, is

$$a_\nu = \frac{3c^2\gamma^2}{32\pi^3\nu^2} \frac{1}{(\nu-\nu_0)^2 + \left(\frac{\gamma}{4\pi}\right)^2}, \quad . \quad . \quad (2.0)$$

where c is the velocity of light and γ is the co-called damping factor, the reciprocal of which is the "mean life time," τ, of the oscillation. In a gas so rarefied that "radiation damping" predominates,

$$\tau = \frac{1}{\gamma} = \frac{3mc^3}{8\pi^2\epsilon^2\nu^2} = 4{\cdot}52\lambda^2, \quad . \quad . \quad (2.1)$$

where m and ϵ are the respective electronic mass and charge and λ is the wave-length corresponding to frequency ν.

Following a suggestion made to me by Prof. J. R. Oppenheimer, the wave-mechanical generalization of (2·0), which, incidentally, is equivalent to that of Weisskopf and Wigner,† is

$$a_\nu = \frac{c^2}{32\pi^3\nu^2} \frac{g_n}{g_m} A_{nm}\Gamma_{nm} \frac{1}{(\nu-\nu_0)^2 + \left(\frac{\Gamma_{nm}}{4\pi}\right)^2}, \quad . \quad (2.2)$$

* *M.N.*, **84**, 354, 1924; **85**, 111, 1924; **86**, 8, 459, 578, 1926; **88**, 188, 493, 1928.

† *Zs. f. Phys.*, **63**, 54, 1930.

where A_{nm} is the Einstein probability of spontaneous transition from energy state n to state m, the weights of which are g_n and g_m respectively. Let γ_{mj} and γ_{nk} be the damping factors for the transitions $m \rightarrow j$ and $n \rightarrow k$. Then, by summing over all relevant states, we get

$$\gamma_m = \sum_j \gamma_{mj} \quad \text{and} \quad \gamma_n = \sum_k \gamma_{nk}. \quad . \quad . \quad (2.30)$$

With these definitions, Γ_{nm}, in (2.3) above, becomes *

$$\Gamma_{nm} = \gamma_m + \gamma_n. \quad . \quad . \quad . \quad . \quad (2.35)$$

We shall make the assumption that the integral of (2.2) over all frequencies is f-times the similar integral of the classical expression (2.0). Solving, we find

$$\int_0^\infty \alpha_\nu d\nu = \frac{c^2}{8\pi\nu^2}\frac{g_n}{g_m}A_{nm} = \frac{3c^2 f}{8\pi\nu^2\tau} = \frac{\pi\epsilon^2}{mc}f, \quad . \quad (2.40)$$

by (2.1), whence:

$$f = \frac{g_n}{g_m}\frac{\tau}{3}A_{nm}, \quad . \quad . \quad . \quad . \quad (2.45)$$

Ladenburg's well-known relation.

Equations (2.0) and (2.2) hold for stationary atoms. When we are concerned with a number of atoms moving at random according to Maxwell's law, it is easily shown † that the atomic absorption coefficient, resulting from the contributions of the individual atoms, each absorbing according to (2.2) but with its resonance frequency shifted to the red or violet by the amount of the Doppler effect, is

$$\alpha_\nu = \frac{c^3}{8\pi^{\frac{3}{2}}\nu^3 v}\frac{g_n}{g_m}A_{nm}e^{-p^2} = \frac{3c^3 f}{8\pi^{\frac{3}{2}}\nu^3 v\tau}e^{-p^2} = 1{\cdot}49 \times 10^{-2}\frac{\lambda}{v}fe^{-p^2}, \quad (2.50)$$

where v is the mean-square velocity defined in the customary way in terms of the temperature T as a parameter, the atomic mass $\overline{m}$, and the Boltzmann constant k,

$$v = \sqrt{\frac{2k\mathrm{T}}{\overline{m}}}. \quad . \quad . \quad . \quad . \quad (2.52)$$

p is defined as follows:—

$$p = \frac{c}{v}\left(\frac{\nu - \nu_0}{\nu_0}\right). \quad . \quad . \quad . \quad . \quad (2.55)$$

Equation (2.50) is approximately valid only so long as $0 \leqslant p < 2.5$, *i.e.*, near the centre of the line. For strict accuracy we must write

$$\alpha_\nu = \frac{3c^3 f}{8\pi^{\frac{3}{2}}\nu^3 v\tau}e^{-p^2} + \frac{3c^4}{32\pi^{\frac{7}{2}}\nu^4 v^2}\frac{f\Gamma_{nm}}{\tau}\mathrm{J}, \quad . \quad . \quad (2.6)$$

where J, a function of p, is given by the following table, computed by quadratures:—

* What we denote by Γ_{nm}, Weisskopf and Wigner denote by $4\pi\Gamma$. The present notation seems to be more consistent.

† Menzel, *L.O. Publ.*, **17** (in press).

TABLE I.

p	$0<p<1\cdot0$	$1\cdot0$	$1\cdot5$	$2\cdot0$	$2\cdot5$	$3\cdot0$	$3\cdot5$
J	*	$2\cdot92$	$1\cdot88$	$0\cdot85$	$0\cdot43$	$0\cdot25$	$0\cdot16$

For $p > 3\cdot5$ we may employ the asymptotic expansion

$$J = \frac{\sqrt{\pi}}{p^2}\left(1 + \frac{3}{2p^2} + \frac{3\cdot5}{4p^4} + \frac{3\cdot5\cdot7}{8p^6} + \ldots\right), \quad . \quad (2.7)$$

which is an excellent approximation for large values of p in spite of the fact that the series is ultimately divergent. For $p>3\cdot5$, however, the first term in (2.6) is usually negligible, and we may write

$$\alpha_\nu = \frac{c^2}{32\pi^3\nu^2}\frac{g_n}{g_m}A_{nm}\Gamma_{nm}\frac{1}{(\nu-\nu_0)^2}\left(1 + \frac{3}{2p^2} + \frac{3\cdot5}{4p^4} + \ldots\right), \quad (2.80)$$

which reduces, in the limiting case of very large p, to

$$\alpha_\nu = \frac{c^2}{32\pi^3\nu^2}\frac{g_n}{g_m}A_{nm}\Gamma_{nm}\frac{1}{(\nu-\nu_0)^2}. \quad . \quad . \quad (2.82)$$

Other equivalent forms are

$$\alpha_\nu = \frac{3c^2}{32\pi^3\nu^2\tau}f\Gamma_{nm}\frac{1}{(\nu-\nu_0)^2} = 7\cdot42\times10^{-25}\lambda^2 f\Gamma_{nm}\frac{\nu_0^2}{(\nu-\nu_0)^2}. \quad (2.85)$$

It is obvious that

$$\frac{\nu_0^2}{(\nu-\nu_0)^2} = \frac{\lambda_0^2}{(\lambda-\lambda_0)^2}, \quad . \quad . \quad . \quad (2.87)$$

which is often more convenient for calculation. Comparison of (2.82) with (2.2) shows that the atomic absorption coefficient in the far wings of a line is unaffected by Doppler motion. It is important to note that equation (2.50) also fulfils the relation (2.40).

The half-width of a spectrum line is defined as the distance $d\nu$ from the line centre at which α_ν reaches one-half of the maximum value. For stationary atoms this occurs at

$$d\nu = \frac{\Gamma_{nm}}{4\pi}, \quad . \quad . \quad . \quad . \quad (2.90)$$

from (2.2), and for moving atoms it occurs at

$$d\nu = \frac{0\cdot83v\nu}{c}. \quad . \quad . \quad . \quad . \quad (2.95)$$

§ 3. *The Flux of Radiation through an Atmosphere in Monochromatic Radiative Equilibrium.*—Milne's analysis of chromospheric equilibrium applies strictly to the ionized calcium atoms therein, which atoms possess ultimate lines not far from the spectrum region of maximum solar energy. In addition, the following postulates are relied upon :—

* Effectively zero.

1. First-stage ionization is complete.
2. Second-stage ionization is inappreciable.
3. The atmosphere is in radiative equilibrium of the monochromatic type.
4. Radiation-pressure plays a predominant rôle in the support of the chromospheric layers.

Since Milne has proved that such refinements as allowing for the doublet character of the atomic levels and the presence of the metastable D-state do not change the order of magnitude of the results or the general character of the equilibrium, I have adopted the simplest view —that the atom possesses but two stationary states. The following analysis is approximately equivalent to Milne's. In a few places I have approached the subject from a slightly different angle with the aim of removing the indefiniteness from certain of Milne's numerical results.

We shall start with the well-known equation of *radiative transfer*, which may be written in the following form :—

$$\cos\theta\frac{dI_\nu}{dx} = -\alpha_\nu n I_\nu + J_\nu. \quad . \quad . \quad . \quad (3.00)$$

This may be derived in the usual manner by considering the gains and losses in a beam of ν-radiation of specific intensity I_ν, making an angle θ with the outward vertical at the point x, as it traverses the layers between x and $x+dx$, x increasing positively outwards. As before, α_ν is the atomic absorption coefficient for ν-radiation, and

$$n = n_1 - \frac{g_1}{g_2}n_2, \quad . \quad . \quad . \quad . \quad (3.01)$$

where n_1 and n_2 are the numbers of atoms in states 1 and 2, and g_1 and g_2 are the statistical weights of the respective states. The negative term in (3.01) represents the " stimulated " emissions, which are here counted as negative absorptions. When the ratio of n_1 to n_2 is given by the Boltzmann expression, *i.e.*, when the state is one of thermodynamic equilibrium, (3.01) becomes

$$n = n_1(1 - e^{-h\nu/kT}). \quad . \quad . \quad . \quad (3.02)$$

Milne states that the results are independent of whether the stimulated emissions are included or not, but an investigation reveals that this is not the case. The point is that the equilibrium depends upon dn/dx just as much as it does upon n, and the former is quite sensitive to the stimulated emissions. J_ν is the specific intensity of the emitted energy, supposed to be independent of θ.

The total energy absorbed is, by definition, integrating over the complete solid angle the element of which is $d\omega$,

$$\alpha_\nu n \int I_\nu d\omega, \quad . \quad . \quad . \quad . \quad (3.03)$$

while the total emitted energy is

$$J_\nu \int d\omega. \quad . \quad . \quad . \quad . \quad (3.04)$$

In equilibrium, these two quantities must be equal. Hence, performing as far as possible the indicated operations:

$$2\pi\alpha_\nu n \int_0^\pi I_\nu \sin\theta d\theta = 2\pi J_\nu \int_0^\pi \sin\theta d\theta = 4\pi J_\nu. \quad . \quad (3.05)$$

Multiply (3.00) by $\sin\theta d\theta$ and integrate.

$$\int_0^\pi \cos\theta \sin\theta \frac{dI_\nu}{dx} d\theta = -\alpha_\nu n \int_0^\pi I_\nu \sin\theta d\theta + 2J_\nu = 0, \quad (3.06)$$

by (3.05). Hence

$$\frac{d}{dx}\int_0^\pi I_\nu \cos\theta \sin\theta d\theta = 0, \quad . \quad . \quad . \quad (3.10)$$

and

$$\int_0^\pi I_\nu \cos\theta \sin\theta d\theta = \text{const.} \quad . \quad . \quad . \quad (3.12)$$

But, since $I_\nu \cos\theta$ is the outward component of I_ν,

$$2\pi \int_0^\pi I_\nu \cos\theta \sin\theta d\theta = \int I_\nu \cos\theta d\omega = \pi F_\nu, \quad . \quad . \quad (3.15)$$

which is, therefore, the net outward flux. As Milne has pointed out, F_ν is the intensity of radiation averaged over the entire solar disc. The derivation proves that *if the equilibrium is of the monochromatic type*, F_ν is a constant with respect to x.

An exact solution of the radiation equations is difficult to achieve. We may, however, proceed after the manner of Schuster and Schwarzschild by assuming, for the intensities of the ingoing and outgoing beams, mean values with respect to θ, I_ν' and I_ν. Equations (3.05) and (3.15) then immediately reduce to

$$2\pi\alpha_\nu n(I_\nu + I_\nu') = 4\pi J_\nu, \quad . \quad . \quad . \quad (3.20)$$

and

$$I_\nu - I_\nu' = F_\nu \quad . \quad . \quad . \quad . \quad (3.25)$$

Multiply (3.00) by $\sin\theta d\theta$ and integrate from 0 to $\pi/2$ and then from $\pi/2$ to π. Thus

$$\left.\begin{aligned} \tfrac{1}{2}\frac{dI_\nu}{dx} &= -\alpha_\nu n I_\nu + J_\nu, \\ \text{and} \\ \tfrac{1}{2}\frac{dI_\nu'}{dx} &= \alpha_\nu n I_\nu' - J_\nu. \end{aligned}\right\} \quad . \quad . \quad . \quad (3.3)$$

Using (3.20) and (3.25) to eliminate I_ν' and J_ν, we find that

$$\frac{dI_\nu}{dx} = \frac{dI_\nu'}{dx} = -\alpha_\nu n F_\nu. \quad . \quad . \quad . \quad (3.4)$$

Now introduce the variable N,

$$N = \int_x n dx. \qquad dN = -n dx. \quad . \quad . \quad (3.5)$$

N may be termed the effective number of *active* atoms above the level x. Milne works in terms of $N a_\nu$, the optical depth, but we shall find the present notation more graphic and better suited to our purpose.

Equations (3.4) integrate to

$$I_\nu = F_\nu(1 + N a_\nu), \quad . \quad . \quad . \quad (3.60)$$

and

$$I_\nu' = F_\nu N a_\nu, \quad . \quad . \quad . \quad . \quad (3.65)$$

on making use of the boundary conditions that $I_\nu \to F_\nu$ and $I_\nu' \to 0$ as $N \to 0$. The total ν-emission, from (3.20), (3.60), and (3.65), is

$$4\pi J_\nu = 2\pi a_\nu n F_\nu(1 + 2N a_\nu). \quad . \quad . \quad . \quad (3.70)$$

The total emission is found by integrating through the absorption line,

$$4\pi \int J_\nu d\nu = 2\pi n \int_0^\infty a_\nu F_\nu(1 + 2N a_\nu) d\nu. \quad . \quad . \quad (3.72)$$

The number of atoms in the higher quantum state is n_2. The total emission per second, therefore, is equal to this multiplied by A_{21}, the Einstein probability of transition from state 2 to state 1, and by $h\nu$, the energy per transition, viz.,

$$n_2 A_{21} h\nu. \quad . \quad . \quad . \quad . \quad (3.75)$$

Equating the identities and solving,

$$\frac{n_2}{n} = \frac{2\pi \int_0^\infty a_\nu F_\nu(1+2N a_\nu) d\nu}{A_{21} h\nu}. \quad . \quad . \quad . \quad (3.80)$$

Up to this point our derivation, although considerably more general, is equivalent to Milne's. In order to obtain Milne's result we make the following approximation :—

$$\frac{n_2}{n} = \frac{2\pi F(1 + 2N a) \int_0^\infty a_\nu d\nu}{A_{21} h\nu}, \quad . \quad . \quad (3.82)$$

where F and a denote mean values of F_ν and a_ν, over the small region $d\nu$. We have already found the general expression for the integral, equation (2.40), whence

$$\frac{n_2}{n} = \frac{g_2}{g_1} \frac{F}{\sigma} (\tfrac{1}{2} + N a), \quad . \quad . \quad . \quad (3.85)$$

where

$$\sigma = \frac{2h\nu^3}{c^2}. \quad . \quad . \quad . \quad . \quad (3.9)$$

Equation (3.85) is identical with Milne's equation (19). It determines the value of n_2/n as a function of N (or of the optical depth).

§ 4. *Mechanical Equilibrium.*—The equation of *mechanical equilibrium* is obtained by balancing the various forces acting on the atom. The energy flux is πF_ν; the total energy absorbed per cm.^{3}, per second, between frequencies ν and $\nu + d\nu$ is $n a_\nu \pi F_\nu d\nu$. Integrating and dividing by c, the velocity of light, the total upward momentum communicated to the volume element, per second, is

$$\frac{\pi n}{c}\int_0^\infty a_\nu F_\nu d\nu. \quad . \quad . \quad . \quad . \quad (4.0)$$

Gravity communicates the downward momentum, per second,

$$(n_1 + n_2)\bar{m}g \quad . \quad . \quad . \quad . \quad (4.1)$$

$\bar{m}$ being the atomic mass and g the acceleration of gravity. The equation of mechanical equilibrium, therefore, is

$$\frac{dp}{dx} = -(n_1 + n_2)\bar{m}g + \frac{\pi n}{c}\int_0^\infty a_\nu F_\nu d\nu. \quad . \quad . \quad (4.2)$$

We approximate, as before, writing

$$\int_0^\infty a_\nu F_\nu d\nu = F\int_0^\infty a_\nu d\nu, \quad . \quad . \quad . \quad (4.3)$$

which gives, on substituting from equation (2.40),

$$\frac{dp}{dx} = -(n_1 + n_2)\bar{m}g + \frac{nc}{8\pi^2}\frac{g_2}{g_1}A_{21}F. \quad . \quad . \quad (4.4)$$

Changing variables by (3.5) and employing (3.85),

$$\frac{dp}{dN} = \bar{m}g\left[1 + \frac{F}{\sigma}(\tfrac{1}{2} + N\alpha)\left(1 + \frac{g_2}{g_1}\right) - \frac{cFA_{21}}{8\pi^2\bar{m}g}\frac{g_2}{g_1}\right], \quad . \quad (4.5)$$

Milne's equation (21). Integrating,

$$p = \bar{m}gN\left[1 + \frac{F}{2\sigma}(1 + N\alpha)\left(1 + \frac{g_2}{g_1}\right) - \frac{cFA_{21}}{8\pi^2\bar{m}g}\frac{g_2}{g_1}\right]. \quad . \quad (4.6)$$

§ 5. *The Density Law for an Atmosphere Completely Supported by Radiation-Pressure.*—If, at the top of the atmosphere, radiation-pressure balances the gravitational attraction, the right-hand side of (4.5) must vanish at great heights, *i.e.*, as $N \to 0$,

$$\lim_{N\to 0}\frac{dp}{dN} = 0 = \bar{m}g\left[1 + \frac{F}{2\sigma}\left(1 + \frac{g_2}{g_1}\right) - \frac{cFA_{21}}{8\pi^2\bar{m}g}\frac{g_2}{g_1}\right], \quad (5.0)$$

whence

$$\frac{F}{\sigma} = \frac{1}{(s - \tfrac{1}{2})\left(1 + \frac{g_2}{g_1}\right)}, \quad . \quad . \quad . \quad (5.10)$$

where

$$s = \frac{A_{21}h\nu}{4c\bar{m}g}. \quad . \quad . \quad . \quad . \quad (5.15)$$

On substituting in (4.6) we get

$$p = \left(1 + \frac{g_2}{g_1}\right)\frac{\bar{m}g\mathrm{N}^2\alpha\mathrm{F}}{2\sigma}. \quad . \quad . \quad . \quad (5.20)$$

We now make the further assumption that the temperature of the atmosphere, *as determined by the molecular velocities*, is sufficiently constant to justify us in assuming a mean value, T_0, for it. This procedure is probably permissible. We have, then, the relation,

$$p = 2(n_1 + n_2)k\mathrm{T}_0, \quad . \quad . \quad . \quad (5.22)$$

since, by hypothesis, there are $2(n_1 + n_2)$ particles (atoms plus electrons) per cm.^{3}. The density, ρ, is given by the expression

$$\frac{\rho}{\bar{m}} = (n_1 + n_2) = \frac{p}{2k\mathrm{T}_0} = \frac{\bar{m}g\mathrm{N}^2\alpha\mathrm{F}}{4\sigma k\mathrm{T}_0}\left(1 + \frac{g_2}{g_1}\right), \quad . \quad (5.25)$$

by (5.20) From (3.85), (5.10), and (5.25) we can determine n as a function of N. Inserting in the following,

$$\frac{d\mathrm{N}}{n} = -dx, \quad . \quad . \quad . \quad . \quad (5.3)$$

and integrating, we obtain

$$\frac{\bar{m}g}{4k\mathrm{T}_0}(x + x_0) = -\log \alpha\mathrm{N} + \frac{1}{\alpha\mathrm{N}}\frac{s}{\left(1 + \frac{g_1}{g_2}\right)}. \quad . \quad (5.4)$$

I have divided the constant of integration into two parts, related to $\log \alpha$ and x_0, in order to get the result into the form of Milne's equation (29).

As Milne points out, F/σ is always very much less than unity, while s is much greater than unity. Taking this into account,

$$\mathrm{F} = \frac{g_1}{g_2}\frac{8\nu^2\bar{m}g}{c\mathrm{A}_{21}}, \quad . \quad . \quad . \quad . \quad (5.50)$$

$$\frac{\rho}{\bar{m}} = \frac{\bar{m}g\alpha}{4k\mathrm{T}_0}\left(1 + \frac{g_1}{g_2}\right)\frac{\mathrm{N}^2}{s}, \quad . \quad . \quad . \quad (5.52)$$

and

$$x + x_0 = \frac{4k\mathrm{T}_0}{\bar{m}g}\frac{s}{\left(1 + \frac{g_1}{g_2}\right)}\frac{1}{\alpha\mathrm{N}}. \quad . \quad . \quad (5.55)$$

Eliminating N between the last two gives

$$\frac{\rho}{\bar{m}} = \frac{4k\mathrm{T}_0}{\alpha\bar{m}g}\frac{s}{\left(1 + \frac{g_1}{g_2}\right)}\frac{1}{(x + x_0)^2}. \quad . \quad . \quad (5.6)$$

Milne evaluates α as follows :—

$$\int_0^\infty \alpha_\nu d\nu = \alpha\Delta\nu = \frac{c^2\mathrm{A}_{21}}{8\pi\nu^2}\frac{g_2}{g_1}, \quad . \quad . \quad . \quad (5.70)$$

by equation (2.40), so that

$$\alpha = \frac{c^2 A_{21}}{8\pi\nu^2}\frac{g_2}{g_1}\frac{1}{\Delta\nu}, \qquad . \quad . \quad . \quad (5.75)$$

which leaves $\Delta\nu$ undetermined. Its value, however, is not arbitrary; one cannot assign to it any value whatever. We should expect it to be about twice the "half-width" as determined by the atomic absorption coefficient, in this case equation (2.95). Thus

$$\Delta\nu \sim \frac{1{\cdot}67v\nu}{c}. \qquad . \quad . \quad . \quad . \quad (5.78)$$

For calcium atoms at 5000° this corresponds to a width $\Delta\lambda$ of about 0·03 A., about thirty times smaller than the value chosen by Milne. We find, from (5.15) and (5.75), that

$$\rho = \frac{4\pi k T_0 \sigma \Delta\nu}{\bar{m}g^2 c(x+x_0)^2}\frac{1}{\left(1+\frac{g_1}{g_2}\right)}, \qquad . \quad . \quad . \quad (5.8)$$

and the mass above level x is

$$\int_x^\infty \rho dx = \frac{4\pi k T_0 \sigma \Delta\nu}{\bar{m}g^2 c(x+x_0)}\frac{1}{\left(1+\frac{g_1}{g_2}\right)}. \qquad . \quad . \quad (5.9)$$

Dividing this by $\bar{m}$, to get the *total* number of atoms above level x, we find that this agrees with N, the effective number of *active* atoms above the same level, by equations (5.15), (5.55), and (5.75). This proves that n_2 is negligible with respect to n_1 for the greater part of the layer under consideration.

§ 6. *More Accurate Formulæ for a "Completely Supported" Atmosphere.*—A better approximation could have been made by evaluating the integral in (3.80) directly. Thus:

$$\frac{n_2}{n_1} = \frac{2\pi F}{A_{21}h\nu}\int_0^\infty \alpha_\nu(1+2N\alpha_\nu)d\nu. \qquad . \quad . \quad (6.0)$$

Introduce equation (2.50) in the following form,

$$\alpha_\nu = \frac{c^3}{8\pi^{\frac{3}{2}}\nu^3 v}\frac{g_2}{g_1}A_{21}e^{-\left[\frac{c}{v}\frac{\nu-\nu_0}{\nu_0}\right]^2}, \qquad . \quad . \quad (6.1)$$

and integrate. This gives, instead of (3.85),

$$\frac{n_2}{n} = \frac{g_2}{g_1}\frac{F}{\sigma}\left(\frac{1}{2}+\frac{hc}{2(2\pi)^{\frac{3}{2}}\sigma v}\frac{g_2}{g_1}A_{21}N\right). \qquad . \quad . \quad (6.2)$$

On repeating the foregoing analysis, we arrive at the following expressions:—

$$\rho = \frac{2(2\pi)^{\frac{3}{2}}kT_0\sigma\nu v}{\bar{m}g^2c^2(x+x_0)^2}\frac{1}{\left(1+\frac{g_1}{g_2}\right)} \qquad . \quad . \quad (6.3)$$

and

$$\int_0^\infty \rho dx = \frac{2(2\pi)^{\frac{3}{2}}kT_0}{\bar{m}g^2c}\frac{\sigma\nu\upsilon}{c}\frac{1}{(x+x_0)}\frac{1}{\left(1+\frac{g_1}{g_2}\right)}, \quad . \quad (6.4)$$

which are consistent with (5.8) and (5.9) if one takes

$$\Delta\nu = \frac{\sqrt{2\pi}\upsilon\nu}{c} = \frac{2\cdot51\nu}{c}\sqrt{\frac{2kT_0}{\bar{m}}}, \quad . \quad . \quad (6.5)$$

by (2.52). Note that this is fairly close to the approximate value, (5.78).*

§ 7. *Numerical Values.* — Assume $T_0 = 5000°$ and take $\nu = 7\cdot5 \times 10^{14}$ corresponding to $\lambda = 4000$ A. For calcium,

$$\bar{m} = 6\cdot6 \times 10^{-23} \quad g = 2\cdot73 \times 10^4 \quad \upsilon = 1\cdot44 \times 10^5,$$

and, considering the ^2P level of Ca^+ as degenerate, $g_2/g_1 = 3$.

Then, numerically, we have

$$\rho = \frac{2\cdot44 \times 10^{-1}}{(x+x_0)^2} \quad . \quad . \quad . \quad . \quad (7.00)$$

and

$$\int_x^\infty \rho dx = \frac{2\cdot44 \times 10^{-1}}{(x+x_0)}. \quad . \quad . \quad . \quad (7.02)$$

The total number of active atoms, N, above level x is

$$N = \frac{\int \rho dx}{\bar{m}} = \frac{3\cdot70 \times 10^{21}}{(x+x_0)}. \quad . \quad . \quad . \quad (7.05)$$

Our next concern is the constant x_0. Milne evaluates it in a manner approximately equivalent to the following. The intensity of the outward beam, as a function of N, is given by (3.60). If we trace it inwards far enough, we eventually come to a point, N, where I_ν has its photospheric value.

Rewriting (3.60), in terms of the average values, F and α,

$$I = F(1 + N\alpha). \quad . \quad . \quad . \quad (7.10)$$

If we take

$$F = rI, \quad . \quad . \quad . \quad . \quad (7.11)$$

where r is the "line blackening" or ratio of the intensity at the line centre to the intensity just outside the line, from (5.50) and Planck's law,

$$I_\nu = \frac{2h\nu^3}{c^2}\frac{1}{(e^{h\nu/kT}-1)}, \quad . \quad . \quad . \quad (7.12)$$

* Unsöld (*Ap. J.*, **69**, 209, 1929) takes $\Delta\lambda = 0\cdot03$ A from observation of the H_3 and K_3 reversals. The significance of his results will be referred to later on.

we get

$$r = \frac{g_1}{g_2}\frac{4c\bar{m}g}{h\nu A_{21}}(e^{h\nu/kT} - 1). \quad . \quad . \quad . \quad (7.13)$$

Also,

$$N = \frac{1 - r}{r\alpha}. \quad . \quad . \quad . \quad . \quad (7.14)$$

But, from (5.70) and (2.95),

$$\alpha = \frac{c^3 A_{21}}{4(2\pi)^{\frac{3}{2}}\nu^3 v}\frac{g_2}{g_1}. \quad . \quad . \quad . \quad . \quad (7.15)$$

Zwaan * has calculated, according to wave mechanics, the transition probability for Ca^+ to be

$$A_{21} = 1{\cdot}55 \times 10^8, \quad . \quad . \quad . \quad (7.20)$$

whence

$$\alpha_\nu = 3{\cdot}27 \times 10^{-12}. \quad . \quad . \quad . \quad (7.25)$$

With the theoretical value of A_{21} and $T = 5740°$, which is correct for the solar radiation, equation (7.13) gives

$$r = 0{\cdot}047. \quad . \quad . \quad . \quad . \quad (7.30)$$

This calls for some comment since it does not agree with the figure Minnaert † obtained observationally, viz.,

$$r = 0{\cdot}07. \quad . \quad . \quad . \quad . \quad (7.35)$$

The preceding analysis makes it clear, however, that r refers not to the mean intensity of the absorption line or even to the mean central intensity, but to the intensity in the narrow central region, $\Delta\nu$, not more than a few hundredths of an angstrom in width. If, for example, the K_3 absorption line is to be attributed to the action of Milne's layer, and the central intensity in that region is 0·047, no contradiction is implied. If, however, the central intensity is 0·07, then radiation-pressure exceeds the gravitational attraction in the ratio 0·07/0·047 = 1·5. Woltjer,‡ in a similar analysis, also employing Zwaan's transition probabilities, finds 1·4 for this ratio. It is obvious, from general considerations, that an atmosphere so constituted would be unstable. Woltjer discusses some of the equilibrium features involved, but in disregarding the presence of the K_3 line he overlooks a very important fact. Unsöld § has measured, upon high-dispersion spectrograms, the central intensities of the H_3 and K_3 lines. He finds an intensity about 0·7 times that of the neighbouring background, which in turn, by Minnaert's work, is 0·07 of the photospheric intensity. This gives, for the K_3 line,

$$r = 0{\cdot}7 \times 0{\cdot}07 = 0{\cdot}049, \quad . \quad . \quad . \quad (7.4)$$

which agrees well within the limits of error with the predicted value above.

* *Arch. Neerlandais*, **12**, 1, 1929.
† *Zs. f. Phys.*, **45**, 610, 1927.
‡ *B.A.N.*, **5**, 43, 1929.
§ *Ap. J.*, **69**, 209, 1929.

Milne reversed the foregoing procedure, substituting the observed value of r (he took $r = 0{\cdot}11$) into (7.13) in order to determine A_{21}. In our case this value of r would give $A_{21} = 0{\cdot}66 \times 10^8$, the slight difference from Milne's value ($0{\cdot}56 \times 10^8$) arising from his taking $T \times 6000°$ instead of 5740°, as we have done.

Combining (7.14), (7.25), and (7.30) we get

$$N = 6{\cdot}20 \times 10^{12} \text{ atoms cm.}^{-2}. \quad . \quad . \quad . \quad (7.5)$$

Milne obtains about 25 times this number because, as I have already noted, of the larger value he provisionally adopted for $\Delta\nu$.

If we take $x = 0$ as the photospheric level, then by eliminating N between (7.05) and (7.5) we find

$$x_0 = 5{\cdot}97 \times 10^8 \text{ cms. or } 5970 \text{ kms.} \quad . \quad . \quad (7.6)$$

Since x_0 is *independent of* $\Delta\nu$ it is not surprising that we obtain a value of the same order of magnitude as Milne's. The density and pressure, at $x = 0$, are

$$\rho_0 = 6{\cdot}85 \times 10^{-19}, \quad . \quad . \quad . \quad (7.7)$$

and

$$p_0 = 1{\cdot}42 \times 10^{-8} \text{ dynes cm.}^{-2} = 1{\cdot}42 \times 10^{-14} \text{ atmos.} \quad (7.8)$$

The partial pressure of calcium atoms would be half this amount or

$$p_{Ca} = 7{\cdot}1 \times 10^{-8} \text{ dynes cm.}^{-2} = 7{\cdot}1 \times 10^{-15} \text{ atmos.} \quad (7.9)$$

§ 8. *The Number of Ionized Calcium Atoms above the Photosphere.*—The relation of pressures in the chromosphere to those in the reversing layer is highly important. A full discussion of all the problems of the photosphere and reversing layer, the production of the continuous spectrum and absorption lines, would carry us too far afield; it is sufficient for our purpose to adopt the conclusions reached by Russell * in his study of the composition of the solar atmosphere. From an investigation of the percentage ionization, as evinced by the relative intensities of neutral and enhanced lines of the same element in the Fraunhofer spectrum, Russell finds the mean pressure of electrons in the reversing layer to be about $2{\cdot}5 \times 10^{-5}$ atmospheres. This corresponds to a temperature of 5740°. The electron-pressure at the base would be twice this amount, or 5×10^{-5} atmospheres.

There has been considerable discussion as to the precise nature of the relation between the contours of spectrum lines and the numbers of atoms above the photosphere, but we shall not concern ourselves here with these problems since all theories agree substantially that the solar atmosphere contains about 2×10^{19} atoms of ionized calcium per cm.^{2} above the point where it becomes practically opaque. I have shown elsewhere † that this number of ionized calcium atoms is consistent with the above value for the electron-pressure, provided the solar abundance of calcium, by volume, is of the order of 5×10^{-4}, which agrees, well within the limits of error, with Russell's result

* *Ap. J.*, **70**, 11, 1929.
† *L.O. Publ.*, **17** (in press).

obtained in an entirely different way. This figure is based upon the assumption that the metallic elements comprise but 1·6 per cent. of the total number of atoms in the solar atmosphere, the remainder of which is for the most part hydrogen. If this percentage is to be altered, our figure for the relative abundance of calcium must be changed proportionally, but the ratio of calcium to all the metallic atoms, 3×10^{-2}, will remain practically the same. Owing to the uncertainty of the correction that must be applied to allow for the probable deviation of the solar atmosphere from thermodynamic equilibrium, the percentage of hydrogen, as given, is subject to revision. It is my opinion, however, based upon a study of the observational data, that if any change is desirable, it is in the direction of an increase rather than a decrease in the proportion of hydrogen.

The partial pressure (in atmospheres) of atoms at the base of an atmosphere with an exponential density distribution in accordance with an *effective* atomic mass, $\bar{m}$, is well known to be

$$p = 10^{-6} N \bar{m} g, \quad . \quad . \quad . \quad . \quad (8.0)$$

where N is the number of atoms in the column. For ionized calcium atoms and electrons in gravitational equilibrium, the effective atomic weight is 20, or half that of the calcium atom. The reduction arises from the electrostatic action of the negative electrons upon the positively charged atoms, as Pannekoek has shown. At the other extreme, we might have an atmosphere so turbulent that the gases would be perfectly mixed, and thus possess a mean atomic weight of the order of unity (it cannot be less, except where hydrogen is largely ionized, in which case it may approach the value 0·5). Hence, the partial pressure of ionized calcium at the base of the reversing layer is defined by the following inequality,

$$10^{-6} < p < 2 \times 10^{-5}. \quad . \quad . \quad . \quad . \quad (8.1)$$

Since the metallic atoms form the major contribution to the electron-pressure, 5×10^{-5} atmospheres, and since calcium forms about 3 per cent. of the metallic atoms, the partial pressure of calcium is probably of the order of $1{\cdot}5 \times 10^{-6}$ atmospheres.

§ 9. *The Low Density at the Base of Milne's Layer—An Objection to Milne's Interpretation.*—When the data just quoted are compared with the results of § 7, a grave difficulty immediately arises. The presence of $6{\cdot}2 \times 10^{12}$ atoms cm.$^{-2}$ is sufficient to reduce the intensity of the emergent beam of radiation to 0·047 its photospheric value, provided the Doppler effect in the upper layers really corresponds to a temperature of 5000°. Yet, in tracing the beam back through these very atoms we arrive, not at the photosphere, which should lie beneath 3×10^6 times as many atoms, but at a level where the partial pressure of calcium is but $7{\cdot}1 \times 10^{-15}$ atmospheres.

We cannot justifiably trace the discrepancy to a possible inapplicability of equation (3.60). As is well known, collisions will modify it, but we should scarcely expect to find them very important at the base of Milne's layer where the *total* pressure probably does not exceed

7×10^{-9} atmospheres, in accord with the extreme assumption that hydrogen is 10^6 times as abundant as calcium, which is 500 times greater than Russell's figures. According to equation (7.14) 2×10^{19} atoms would reduce the intensity to

$$r = 5 \times 10^{-9}, \quad . \quad . \quad . \quad . \quad (9.0)$$

a value negligibly small, and certainly far from the observed figure. This is a well-known defect of the Schuster approximation, which should hold to a fair degree of accuracy for the intensity of an absorption line produced by a quantity of *cool rarefied* gas placed in front of a radiating surface, but which fails in hot stellar atmospheres.*

I must hasten to say that Milne is perfectly aware that a difficulty exists, and he suggests the following solution of the problem. Schwarzschild has shown that, if the effective temperature of the photosphere is T_1, the effective temperature of the boundary, T′, should approximately obey the relation

$$T' = 2^{-\frac{1}{4}}T_1. \quad . \quad . \quad . \quad . \quad (9.1)$$

Putting $T_1 = 5740°$, we get $T' = 4830°$. The argument is that the radiation from the centres of the lines comes from atoms near the boundary. Hence the intensity of the line centre should correspond to black-body radiation at the boundary temperature. Something of the sort undoubtedly occurs, even though the conception of boundary is extremely elusive. It is fairly obvious, however, that below the boundary collisions must be sufficiently frequent to supply ν-radiation through collisional excitation, while above the boundary collisions must be relatively rare. These are the layers, referred to in § 1, to which Milne assigns the respective names *reversing layer* and *chromosphere*. It seems to me that it would have been better to have first made sure whether any one of Milne's arbitrary atmospheric layers corresponded to the chromosphere we observe. For that reason I have specifically avoided the use of the word chromosphere, wherever I could, and substituted the designation *Milne's layer*, until the theoretical discussion could be tied up to observation. If, however, we tentatively accept Milne's definition, one slight modification is necessary. As Milne points out, we should trace the beam back to the boundary layer instead of to the photosphere itself. Let

$$F = rI = r_0 I_0 \quad . \quad . \quad . \quad . \quad (9.2)$$

where I_0 is the intensity of emission at the boundary layer. Then

$$F = \frac{2h\nu^3}{c^2} r e^{-h\nu/kT_1} = \frac{2h\nu^3}{c^2} r_0 e^{-h\nu/kT'}, \quad . \quad . \quad . \quad (9.3)$$

(the Wien approximation is amply accurate) whence

$$r_0 = r e^{\frac{h\nu}{kT_1}\left(\frac{T_1}{T'}-1\right)} = 3{\cdot}3r = 0{\cdot}15, \quad . \quad . \quad (9.40)$$

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* Stewart, *M.N.*, **85**, 734, 1925.

for $$r = 0{\cdot}047 \quad . \quad . \quad . \quad . \quad (9.45)$$

Putting r_0, instead of r, into equation (7.14), we get

$$N = 1{\cdot}66 \times 10^{12}, \quad . \quad . \quad . \quad (9.5)$$

whence

$$x_0 = 2{\cdot}22 \times 10^9 \text{ cms.} = 2{\cdot}22 \times 10^4 \text{ kms.} \quad . \quad (9.6)$$

$$\rho_0 = 4{\cdot}9 \times 10^{-20} \text{ gm. cm.}^{-3} \quad . \quad . \quad . \quad (9.7)$$

$$p_0 = 1{\cdot}0 \times 10^{-9} \text{ dynes cm.}^{-2} = 1{\cdot}0 \times 10^{-15} \text{ atmos.} \quad (9.8)$$

$$p_{Ca} = 5{\cdot}0 \times 10^{-16} \text{ atmos.}$$

$$\rho = \frac{2{\cdot}44 \times 10^{-1}}{(x + 2{\cdot}22 \times 10^9)^2} \quad . \quad . \quad . \quad (9.9)$$

The density gradient on this theory comes out absurdly low. To put it in another way, we should expect to observe the calcium chromosphere to much higher levels than we actually do. For example, the density at the oft-quoted height, 14,000 kms., would be only about

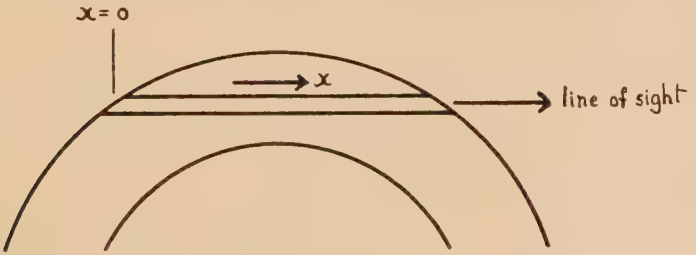

Fig. 1.

one-third of the density at the base. This, in itself, would constitute considerable objection to the theory, even if it were not so difficult to reconcile the low pressures with Milne's assumption that "local thermodynamic equilibrium" sets in at this point.

§ 10. *The Energy Emitted by a Fully Supported Chromosphere.*—If there were no "self-reversal" present, the intensity of the chromosphere could be simply calculated by integrating the total light emitted by all the atoms composing it. Actually the energy from the far side of the atmosphere suffers considerable absorption in traversing the atoms that lie between the point of emission and the nearer edge of the chromosphere, and we must allow for this effect.

We shall take our co-ordinate system as indicated in fig. 1, x being 0 at the farther edge of the chromosphere. The beam of radiation of intensity, I_ν, flowing in the positive direction of the x-axis, suffers a decrement of $a_\nu n I_\nu dx$ over the distance dx, but it gains, by emission, the amount $J_\nu dx$; hence the equation of transfer is

$$\frac{dI_\nu}{dx} = -a_\nu n I_\nu + J_\nu \quad . \quad . \quad . \quad (10.00)$$

where the various quantities have the same significance as in equation (3.00).

The integral of this equation at the point x_1 is

$$I_\nu = e^{-\int_{x_1}^{x_1} \alpha_\nu n dx}\left[A + \int e^{\int_{x_1}^{x} \alpha_\nu n dx} J_\nu dx\right], \quad . \quad . \quad (10.05)$$

where A is a constant of integration to be evaluated from the condition that

$$I_\nu \longrightarrow 0 \quad \text{as} \quad x_1 \longrightarrow 0.$$

Make this substitution and change variables as follows :—

$$N = \int_x^\infty n dx \;, \quad dN = -n dx \quad . \quad . \quad (10.10)$$

$$N_1 = \int_{x_1}^\infty n dx = -\int^{x_1} n dx \quad . \quad . \quad . \quad (10.12)$$

$$N_{0\infty} = \int_0^\infty n dx. \quad . \quad . \quad . \quad . \quad . \quad (10.15)$$

Then, after a little algebra, we get

$$I_\nu = \int_{N_1}^{N_{0\infty}} e^{-\alpha_\nu (N - N_1)} \frac{J_\nu}{n} dN. \quad . \quad . \quad . \quad (10.20)$$

Hence the emergent intensity at $N = 0$ is

$$I_\nu^* = \int_0^{N_{0\infty}} e^{-\alpha_\nu N} \frac{J_\nu}{n} dN. \quad . \quad . \quad . \quad (10.25)$$

To effect integration of this equation we must know how J_ν varies with N. This has already been calculated in equation (3.70), which may be written as follows :—

$$J_\nu = \frac{\alpha_\nu n F_\nu}{2}(1 + 2\bar{N}\alpha_\nu). \quad . \quad . \quad . \quad (10.30)$$

We must not confuse $\bar{N}$ with the N of our present section. The former is now used to indicate the number of atoms measured vertically, *i.e.* along a solar radius, down to a given point. The latter is used to indicate the number of atoms measured along a line tangential to the sun, down to the same point. N is a function of $\bar{N}$ and can be computed once the density distribution is known. In the present instance the integration can be effected only numerically.

We can, however, considerably simplify the procedure by introducing a mean value for J_ν. This will not cause any great error since J_ν can be neither less than the emission from the outer boundary nor greater than half of the flux from the inner (the Schwarzschild) boundary. The former corresponds to $\bar{N}\alpha_\nu = 0$, the latter to $\bar{N}\alpha_\nu = 5{\cdot}7$. Thus

$$\frac{\alpha_\nu n F_\nu}{2} < J_\nu < \frac{\alpha_\nu n F_\nu}{2}\,(12.4). \quad . \quad . \quad . \quad (10.32)$$

Striking an approximate mean, we may take

$$J_\nu \sim 5\alpha_\nu n F_\nu. \quad . \quad . \quad . \quad . \quad (10.35)$$

Even this value will be too large in the wings of the lines since there $\overline{N}\alpha_\nu << 5{\cdot}7$, but we shall be safe in assuming that, however inaccurate (10.35) may be, it will not give us an overestimate of the magnitude of chromospheric emission. Substituting this expression into equation (10.25) and integrating, we find

$$I_\nu^* = 5F_\nu(1 - e^{-\alpha_\nu N_{0y}}), \quad . \quad . \quad . \quad (10.40)$$

i.e., for large optical depths,

$$I_\nu^* \sim 5F_\nu, \quad . \quad . \quad . \quad . \quad (10.42)$$

while for a transparent gas

$$I_\nu^* \sim 5\alpha_\nu N_{0y} F_\nu. \quad . \quad . \quad . \quad . \quad (10.45)$$

A strong chromospheric line will tend to be "flat-topped" and of approximate intensity $5F_\nu$ over the spectral region where

$$\alpha_\nu N_{0y} \geqslant \tfrac{1}{2}, \quad . \quad . \quad . \quad . \quad (10.48)$$

say, α_ν being, of course, a function of $\nu - \nu_0$. Beyond the point $\nu - \nu_0$ defined by (10.48) the energy rapidly declines, so that we may write, approximately, for the total energy in an emission line,

$$I^* = 2\int_{\nu_0}^{\infty} I_\nu^* d\nu \sim 5F_\nu \Delta\nu. \quad . \quad . \quad . \quad (10.5)$$

$\Delta\nu$ is the effective width of the emission line. Referring to fig. 2, we

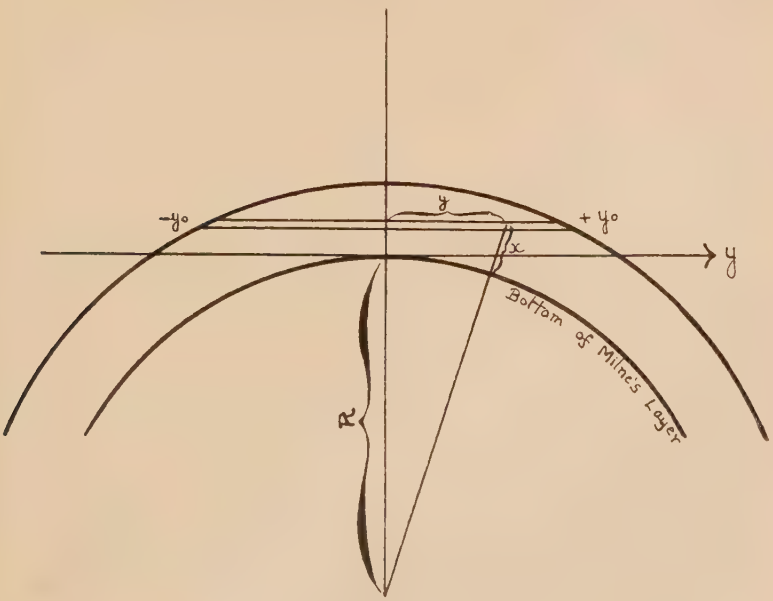

Fig. 2.

see that N_{0y}, the number of atoms lying in the column $-y_0$ to $+y_0$, one cm.^{2} in cross-section, is given by the integral

$$N_{0y} = 2\int_0^\infty \frac{\rho}{\bar{m}} dy. \quad . \quad . \quad . \quad (10.60)$$

Also we have, approximately,

$$y^2 = 2Rx, \quad . \quad . \quad . \quad . \quad (10.62)$$

R being the solar radius.

Introducing ρ from (9.9), and integrating,

$$N_{0y} = 1{\cdot}3 \times 10^{13} \text{ atoms cm.}^{-2} \quad . \quad . \quad (10{\cdot}65)$$

In so rarefied a gas we can legitimately assume predominance of radiational transitions in calculating the mean lifetime of the atomic states. Near the line centres we can adopt (2.50), which becomes, numerically, for the K-line,

$$a_\nu = 2{\cdot}92 \times 10^{-12} e^{-p^2}, \qquad p = 52(\lambda - \lambda_0), \quad . \quad (10.70)$$

when $\lambda - \lambda_0$ is in angstroms, provided also that $0 < (\lambda - \lambda_0) < 0{\cdot}05$. For $(\lambda - \lambda_0) > 0{\cdot}07$, we have, from equation (2.7),

$$a_\nu \sim \frac{2{\cdot}2 \times 10^{-18}}{(\lambda - \lambda_0)^2}\left(1 + \frac{3}{2p^2} + \frac{3{\cdot}5}{4p^4} + \ldots\right). \quad . \quad (10.75)$$

By (10.48), we are to find what value of $\lambda - \lambda_0$ corresponds to $a_\nu = 2{\cdot}4 \times 10^{-14}$. This requires the use of equation (10.70). It is easily found that

$$\lambda - \lambda_0 \sim 0{\cdot}042. \quad . \quad . \quad . \quad . \quad (10.80)$$

The total energy emitted by one cm.^{2} of the projected chromosphere is

$$\pi I^* = 5\pi F_\nu \Delta\nu = 1{\cdot}66 \times 10^5 \text{ ergs cm.}^{-2}, \quad . \quad (10.82)$$

from (9.3) and (9.45) with $T_1 = 5740°$, and

$$\Delta\nu = 2 \times 10^{-8} \frac{c}{\lambda^2}(\lambda - \lambda_0).$$

Though I have not been able to locate any reliable experimental determinations of this quantity, we know this much, at least. Pannekoek and Minnaert † find that the emission of the Hγ line near the base of the chromosphere is

$$2 \times 10^6 \text{ ergs cm.}^{-2} \text{ sec.}^{-1}.‡ \quad . \quad . \quad . \quad (10.85)$$

Furthermore, we know that K is considerably more intense than Hγ. A comparison of (10.85) and (10.82) is convincing proof that Milne's layer, if it exists at all, does not begin until heights such that the intensity of emission has fallen to not more than a few per cent. of the intensity at the base of the observed chromosphere. Woltjer,§ who

† *Verh. der Kon. Akad. v. Wetenschappen te Amsterdam*, **13**, 1, 1928.

‡ Note that the factor π has been included (numerically) to give total emission, not the intensity per unit solid angle.

§ *B.A.N.*, **4**, 259, 1928.

came to the reverse conclusion, that the intensity of the chromosphere according to Milne's theory is greater than that observed, neglected, as Pannekoek has pointed out,† to take into consideration the extreme narrowness of the chromospheric line.

The total emission per cm.^{2} of the chromosphere, had we not allowed for absorption, may be computed.

$$\pi I^* = \pi N_{0y} I_\nu \int_0^\infty \alpha_\nu d\nu = \pi N_{0y} I_\nu \frac{c^2}{8\pi\nu^2} \frac{g_2}{g_1} A_{21}, \quad . \quad (10.90)$$

by (2.40).

For I_ν we must introduce a sort of mean value. According to (3.60) and (3.65), I_ν varies in the chromosphere by

$$I_\nu = \frac{I_\nu + I_\nu'}{2} = \frac{F_\nu}{2}(1 + 2N\alpha_\nu) \quad . \quad . \quad (10.92)$$

i.e.,

$$\frac{F_\nu}{2} < I_\nu < 12{\cdot}4 \frac{F_\nu}{2}, \quad . \quad . \quad . \quad (10.94)$$

by the same argument that led to (10.32).

A mean value of

$$I_\nu = 5F_\nu \quad . \quad . \quad . \quad . \quad . \quad (10.96)$$

would be consistent with our earlier approximation (10.35). Introducing (9.3),

$$\pi I^* = \tfrac{5}{4} N_{0y} r \frac{g_2}{g_1} e^{-h\nu/kT_1} A_{21} h\nu = 4{\cdot}1 \times 10^6, \quad . \quad (10.98)$$

which is twenty-five times greater than the intensity when self-reversal is taken into account.

These various considerations appear to be conclusive arguments against the validity of Milne's first hypothesis, that the chromosphere is " fully supported " by radiation-pressure.

§ 11. *The Density Law of an Atmosphere partially supported by Radiation-Pressure.*—We next consider the case where radiation-pressure supports, at high atmospheric levels, not the entire weight of the atom, but the fraction $1 - \mu$ ($0 < \mu < 1$). Though the analysis is not so interesting from the mathematical viewpoint as that of the totally supported atmosphere, it is physically more important. Consequently I have extended, somewhat, Milne's treatment of the problem.

First of all, since n exhibits so small a variation, we shall regard it as constant in (4.4). In other words, we shall neglect n_2 with respect to n_1. With this simplification,

$$\frac{dp}{dx} = -n\bar{m}g\left(1 - \frac{c}{8\nu^2\bar{m}g} \frac{g_2}{g_1} A_{21} F\right), \quad . \quad . \quad (11.0)$$

since

$$n \sim n_1 + n_2 \sim n_1 - \frac{g_1}{g_2} n_2. \quad . \quad . \quad . \quad (11.05)$$

† *B.A.N.*, **4**, 263, 1928.

According to the foregoing assumption, the coefficient of $-n\bar{m}g$ takes on the value μ, *i.e.*,

$$F = (1-\mu)\frac{g_1}{g_2}\frac{8\nu^2\bar{m}g}{cA_{21}} \quad . \quad . \quad . \quad (11.1)$$

and

$$\frac{dp}{dx} = -n\mu\bar{m}g \quad . \quad . \quad . \quad . \quad (11.2)$$

provided μ is not too near zero.

On dividing through by (5.22) and integrating,

$$p = p_0 e^{-\frac{\mu\bar{m}gx}{2kT_0}}. \quad . \quad . \quad . \quad (11.30)$$

$$\rho = \rho_0 e^{-\frac{\mu\bar{m}gx}{2kT_0}}. \quad . \quad . \quad . \quad (11.32)$$

$$\int_0^\infty \rho dx = \frac{2\rho_0 kT_0}{\mu\bar{m}g} e^{-\frac{\mu\bar{m}gx}{2kT_0}} \quad . \quad . \quad . \quad (11.34)$$

$$N = \frac{1}{m}\int_x^\infty \rho dx = \frac{2\rho_0 kT_0}{\mu\bar{m}^2 g} e^{\frac{\mu\bar{m}gx}{2kT_0}} \quad . \quad . \quad (11.36)$$

These agree with Milne's results. The number of atoms above $x = 0$ is

$$N_0 = \frac{\rho_0 kT_0}{\mu\bar{m}^2 g} = 1{\cdot}16 \times 10^{28}\rho_0/\mu. \quad . \quad . \quad (11.4)$$

If, as before, we identify the level $x = 0$ with the aforementioned boundary layer, at temperature T', we must adopt, for N_0, a value consistent with the equation of radiative flux (3.60), or its equivalent (7.14).

By equations (7.14), (9.3), and (11.1), the equation (11.4) becomes

$$\frac{1-r_0}{r_0\alpha} = 1{\cdot}16 \times 10^{28}\rho_0 \Big/ \Big(1 - \frac{h\nu r_0}{4\bar{m}gc}\frac{g_2}{g_1}A_{21}e^{-h\nu/kT'}\Big). \quad . \quad (11.5)$$

Numerically, then, employing (7.25) and (9.40),

$$\rho_0 = 2{\cdot}64 \times 10^{-17}\frac{(1-r_0)(1-6{\cdot}4r_0)}{r_0}$$

$$= 8{\cdot}00 \times 10^{-18}\frac{(1-3.3r)(1-21{\cdot}2r)}{r}. \quad (11.6)$$

For $r = 0$, ρ_0 becomes infinite ; this occurs because an infinite amount of material is needed to reduce the central intensity to zero. This extreme, of course, is physically impossible for, as soon as the density reaches a certain value, monochromatic radiative equilibrium cannot exist and the equations fail. On the other hand, ρ_0 becomes 0 for $r = 0{\cdot}047$, *i.e.*, when radiation and gravity balance. I have already pointed out that the equations of " partial support " cannot be applied in this case, which requires the special treatment of §§ 3 and 4. Equili-

brium of a "fully supported" atmosphere is determined chiefly by small variations in n or n_2, which we have here assumed to be constant.

The following table of ρ_0, ρ_{Ca}, and N_0 as functions of r and μ is of particular interest:

	0*	0·001	0·01	0·1	0·2	0·5	0·9
	0·0472*	0·0471	0·0466	0·0425	0·0377	0·0236	0·00472
	$4·9 \times 10^{-20}$*	$1·4 \times 10^{-19}$	$1·4 \times 10^{-18}$	$1·6 \times 10^{-17}$	$3·7 \times 10^{-17}$	$1·5 \times 10^{-16}$	$1·5 \times 10^{-15}$
(atm.)	$5·0 \times 10^{-16}$*	$1·4 \times 10^{-15}$	$1·4 \times 10^{-14}$	$1·6 \times 10^{-13}$	$3·8 \times 10^{-13}$	$1·5 \times 10^{-12}$	$1·5 \times 10^{-11}$
	$1·66 \times 10^{12}$*	$1·66 \times 10^{12}$	$1·68 \times 10^{12}$	$1·88 \times 10^{12}$	$2·16 \times 10^{12}$	$3·57 \times 10^{12}$	$1·94 \times 10^{13}$

* Fully supported atmosphere.

If radiation-pressure is to be of any great help in accounting for the probably low chromospheric density gradient, the coefficient of support, μ, must certainly be less than 0·5. In fact, it probably should be smaller than 0·1. This requires that the partial pressure of ionized calcium atoms in the chromosphere be less than about 2×10^{-13} atmospheres, *i.e.*, at least 10^7 times smaller than the pressure of ionized calcium at the base of the reversing layer, equation (8.0). We shall see presently that this figure does not agree with observational data, but we can see at once that it is inconsistent with the assumptions upon which the theory was based. By definition N_0 is the number of atoms lying above the layer in "local thermodynamic equilibrium." Yet the partial pressure found for the interface is extremely small. If, as Milne believes, calcium is a predominant constituent of the solar atmosphere, the total pressure at the base of Milne's layer is still extremely small, which belies the implied requirement that collisions be important below the interface. It seems that we are obliged to postulate a high abundance of other elements. Milne has questioned * the existence of an excess of hydrogen, but, if the foregoing argument is correct, it would appear that his theory demands it.

Unfortunately, difficulties remain even if the presence of considerable hydrogen is assumed. Hydrogen, in the normal state, can absorb energy only in the extreme ultra-violet where, if the sun radiates like a black body at temperature around 6000°, there will be insufficient energy to give rise to appreciable radiation-pressure. The presence of an abundance of hydrogen will tend to change the character of the equilibrium, unless the atmosphere is so quiescent that each type of atom can attain equilibrium independently, according to some form of Dalton's law. The existence of sun-spot vortices, prominences, etc., must be taken into account. These will tend to keep the atmosphere "stirred up." We are thus led, quite naturally, to consider a chromosphere consisting chiefly of hydrogen and one in which turbulence may play an important part. This problem has been studied by McCrea.†

It is of interest to evaluate the emission of a partially supported chromosphere. The total number of atoms, $N_{0\prime\prime}$, in cm.^{2} column of

* *Obs.*, **53**, 119, 1930.
† *M.N.*, **89**, 483, 718, 1929.

chromosphere viewed transversely has been computed according to equations (10.60), (10.62), and (11.32), yielding the result:

$$N_{0y} = 2\sqrt{\frac{\pi R k T_0}{\mu \bar{m} g}}\frac{\rho_0}{\bar{m}} = 8{\cdot}7 \times 10^{30}\frac{\rho_0}{\sqrt{\mu}}. \quad . \quad (11.7)$$

Substituting from the previous table, we get, for the respective values of μ:

μ	0·001	0·01	0·1	0·2	0·5	0·9
N_{0y}	$3{\cdot}9\times10^{13}$	$1{\cdot}2\times10^{14}$	$4{\cdot}4\times10^{14}$	$7{\cdot}2\times10^{14}$	$1{\cdot}8\times10^{15}$	$1{\cdot}4\times10^{16}$

For $\mu = 10^{-1}$ we find, employing (10.48) and (2.6), that

$$\lambda - \lambda_0 \sim 0{\cdot}05,$$

and, analogous to (10.82),

$$\pi I^* = 2{\cdot}0 \times 10^5 \text{ ergs cm.}^{-2}. \quad . \quad . \quad (11.8)$$

The difficulty encountered in Milne's theory of the totally supported chromosphere, viz. that it predicts too weak an emission, thus carries over to the theory of the partially supported atmosphere and constitutes a fairly strong argument against the identity of Milne's layer with the observed chromosphere. Broadening of the lines by turbulent motion will increase the emission, but turbulence is not part of Milne's theory, which treats of an atmosphere essentially static. The turbulence of the observed chromosphere is more fully discussed in a forthcoming volume of the *Publications of the Lick Observatory*.

§ 12. *Pressures in the Observed Chromosphere.*—Russell,* analyzing the differences in intensities between the solar lines produced by the same atom in its neutral and ionized states, was able to determine, from the dissociation formula, the electron-pressure in the reversing layer. Menzel,† in a parallel analysis of the flash spectrum, found the pressure P of electrons at the base of the chromosphere to be about 10^{-7} atmospheres. For the most part, the electrons come from ionization of hydrogen and metallic atoms. Thus

$$P = 10^{-6}(n_H x_H + n_M x_M)kT, \quad . \quad . \quad . \quad (12.0)$$

where n_H and n_M are the numbers of hydrogen and metallic atoms and x_H and x_M their respective percentage ionizations. If the chromosphere were in thermodynamic equilibrium, at a temperature of some 4700°, x_H would be of the order of 2×10^{-5}. The chromosphere, however, is far from being in thermodynamic equilibrium (witness the strength of the lines of *H*, *He*, and He^+) and it is not improbable that hydrogen is much more dissociated than the simple theory predicts. If we write

$$x_H = 2 \times 10^{-3}, \quad . \quad . \quad . \quad . \quad (12.1)$$

we shall probably be not far wrong. We may take, unambiguously,

$$x_M = 1. \quad . \quad . \quad . \quad . \quad (12.2)$$

* *Ap. J.*, **70**, 11, 1929. † *Publ. Lick Obs.*, **17** (in press).

Accordingly, equation (12.0) becomes

$$n_M kT = \frac{P}{10^{-6}} \frac{1}{\left(\frac{n_H}{n_M} x_H + 1\right)}. \quad . \quad . \quad . \quad (12.3)$$

The partial pressure of calcium is

$$p_{Ca} = 10^{-6} a_{Ca} n_M kT = \frac{P a_{Ca}}{\left(\frac{n_H}{n_M} x_H + 1\right)}, \quad . \quad . \quad (12.4)$$

where a_{Ca} is the proportion of calcium by volume among the metals. According to Russell's figures,

$$a_{Ca} = 3 \times 10^{-2}. \quad . \quad . \quad . \quad (12.50)$$

The indicated chromospheric value is

$$a_{Ca} = 6 \times 10^{-2} \quad . \quad . \quad . \quad . \quad (12.55)$$

Russell found

$$\frac{n_H}{n_M} \sim 2{\cdot}5 \times 10^3, \quad . \quad . \quad . \quad . \quad (12.6)$$

though he later modified this to

$$\frac{n_H}{n_M} \sim 60 \quad . \quad . \quad . \quad . \quad (12.7)$$

in accord with an assumed mean atomic weight of 2 for the solar atmosphere. The ratio found for the chromosphere is much larger than either of these figures:

$$\frac{n_H}{n_M} \sim 6 \times 10^5. \quad . \quad . \quad . \quad . \quad (12.8)$$

It is probably safer to choose the last value, since it is derived specifically for the chromosphere. Thus, employing (12.4), (12.55), and (12.8), we find

$$p_{Ca} \sim 5 \times 10^{-12} \text{ atmospheres}, \quad . \quad . \quad (12.9)$$

which, although somewhat greater than the theoretical pressures predicted by Milne's theories, is sufficiently near them to admit of the possibility of radiation-pressure playing some part in the support of chromospheric calcium, especially at high levels.

It is to be noted, however, that this possibility is dependent upon the reality of a tremendous excess of hydrogen and an appreciable deviation from thermodynamic equilibrium. If we reduce the proportion of hydrogen even to that indicated by Russell for the reversing layer, the pressure of calcium is correspondingly raised to a point where radiation-pressure becomes totally inadequate to act as an agency of support. It seems probable that we can identify Milne's layer only with the uppermost chromospheric layers. The question of chromospheric brightness is a deciding factor in the case. A discussion of the

effect of the predominance of hydrogen and of turbulent motion of the gases upon the character of chromospheric equilibrium does not lie within the scope of this paper. The reader is referred to Volume 17 of the *Publications of the Lick Observatory*, where certain points are further discussed.

Summary.

Milne's theories of chromospheric equilibrium have been subjected to a quantitative study. The quantity, $\Delta\nu$, the effective width of the absorption line, which Milne left undetermined, has been evaluated explicitly. The removal of this indefiniteness has afforded an exact determination of what the partial pressure of calcium atoms would be at the base of an atmosphere "fully supported" by radiation-pressure. The value derived is 5×10^{-16} atmospheres. Milne has argued that the distinction between the chromosphere and reversing layer is largely a matter of collisions, which he supposes to be relatively rare in the former and relatively frequent in the latter. The above low value of the partial pressure of calcium is incompatible with the assumption that "local thermodynamic equilibrium" with its predominant collisions begins at this pressure. The density gradient of the "fully supported" atmosphere comes out absurdly low, the calculated density at the oft-quoted height, 14,000 km., being only about one-third of the density at the base. Also, the intensity of light emitted from the base of such an atmosphere would be at least ten times less than that emitted from the base of the observed chromosphere. These facts form weighty objections to the view that the chromosphere is being fully supported by radiation-pressure.

Unfortunately, much the same objections can be urged against the theory of a partially supported atmosphere. For an atmosphere where nine-tenths of the weight of the atom is supported by radiation-pressure, the partial pressure of calcium atoms at the base would be $1{\cdot}6 \times 10^{-13}$ atmospheres. If hydrogen atoms were 10^7 times as abundant as calcium atoms, the total pressure at the base would be of the order of 10^{-5} atmospheres, which might possibly be high enough to permit incipient collisional action. Without this tremendous excess of hydrogen, Milne's theory is unacceptable. It seems likely, however, that the observed chromosphere corresponds to a somewhat lower level than the one Milne's theory refers to. Milne's layer, if it exists at all, should be identified with the uppermost part of the chromosphere.

It is a pleasure to record my indebtedness to Dr. Henry Norris Russell, who read this paper in its initial stages and who made numerous valuable suggestions. I also wish to express my appreciation to Dr. J. R. Oppenheimer, with whom I have frequently discussed questions dealing with the absorption and emission of light by atoms.

*On Emden's Differential Equation.** By Eberhard Hopf.

§ 1. *Introduction.*—Emden's equation

$$\frac{1}{\xi^2}\frac{d}{d\xi}\left(\xi^2\frac{d\theta}{d\xi}\right)+\theta^n=0 \qquad . \quad . \quad . \quad (1)$$

is well known to play a fundamental rôle in modern theories on internal constitution of the stars. Eddington's ideal gas sphere, in equilibrium under gravitation, gas and radiation pressure, is under natural assumptions governed by (1), $n = 3$. Likewise Milne's new model of a sphere of degenerate gas in the sense of Fermi-Dirac statistics has a structure governed by (1), $n = \frac{3}{2}$. In each case ξ represents a "reduced" distance from the centre, θ^n a "reduced" density. Eddington's star model is governed by the so-called Emden-solutions of (1) leading to a finite central density. In his recent work on stellar structure Milne,† however, makes astrophysical applications of different kinds of solutions of Emden's equation. In connection with this, R. H. Fowler ‡ gave a complete and rigorous analysis of the solutions of (1), $1 < n \leqq 3$, and of more general equations. However, the proofs constructed by Fowler can be simplified to a large extent. In this paper I intend to give shorter proofs of Fowler's important results on the arrangement and the asymptotic behaviour of the solutions for $\xi \to 0$.

We consider, like Fowler, the more general equation

$$\frac{1}{\xi^\lambda}\frac{d}{d\xi}\left(\xi^2\frac{d\theta}{d\xi}\right)+\theta^n=0\,\S\,; \quad \lambda>0,\ n>1. \quad . \quad . \quad (2)$$

The number

$$\mu=\frac{\lambda}{n-1} \qquad . \quad . \quad . \quad . \quad . \quad (3)$$

plays an important rôle in the theory of the equation (2). For instance, a given solution $\theta(\xi)$ implies a whole family $A^\mu\theta(A\xi)$ of solutions. Closely connected with this well-known "homology group" is the reduction of (2) to a single equation of first order, obtained already by Emden. Putting $x = \frac{1}{\xi}$ we obtain

$$\frac{d^2\theta}{dx^2}+x^{-\lambda-2}\theta^n=0, \quad x=\frac{1}{\xi}. \quad . \quad . \quad . \quad (2a)$$

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* A second paper was reserved for applications to Milne's centrally condensed stars. When this paper was received by the R.A.S., three papers by A. S. Eddington, B. Strömgren, and T. A. Cowling, containing the same result, were already going to be printed. As my paper mentioned agrees closely with the paper by Cowling, a publication seemed to me to be useless.

† E. A. Milne, *M.N.*, **91**, 4 (1930).

‡ R. H. Fowler, *M.N.*, **91**, 63 (1930). See this paper for references to earlier mathematical work.

§ λ is connected with the σ introduced by Fowler by $\lambda + \sigma + 2 = 0$.

The choice of new variables t, u defined by

$$\xi = \frac{1}{x} = e^{-t}, \quad \theta = e^{\mu t}u = \frac{u}{\xi^{\mu}} = x^{\mu}u \quad . \quad . \quad (4)$$

leads to the form $\left(\dot{u} \equiv \frac{du}{dt}\right)$

$$\ddot{u} + (2\mu - 1)\dot{u} + \mu(\mu - 1)u + u^n = 0 \quad . \quad . \quad (2b)$$

of the equation (2). Putting $\dot{u} = v$, we obtain the system of equations of first order

$$\left.\begin{aligned} \dot{u} &= v \\ \dot{v} &= -(2\mu - 1)v - \mu(\mu - 1)u - u^n \end{aligned}\right\} \quad . \quad . \quad (2c)$$

and the single equation of first order

$$\frac{dv}{du} = -(2\mu - 1) - \frac{\mu(\mu - 1)u + u^n}{v}. \quad . \quad . \quad (2d)$$

In this paper, like Fowler, we only treat the case

$$\mu \geqq 1, \quad n > 1 \quad . \quad . \quad . \quad . \quad (5)$$

i.e. $n - \lambda - 1 \leqq 0$. In the case of Emden's equation ($\lambda = 2$) this is equivalent to $1 < n \leqq 3$, including the astrophysically important cases, $n = 3$, $n = \frac{5}{2}$.

§ 2. *Relative Arrangement of the Solutions. E-curves.*—We consider the solutions of the system (2c) or the equivalent equation (2d) as far as $u \geqq 0$, *i.e.* as far as it is of direct astrophysical interest. The solutions $u(t)$, $v(t)$ may be represented by curves in the right half ($u \geqq 0$) of the u-v-plane. On account of $\dot{u} = v$ each piece of such a curve lying wholly in the half-plane $v > 0$ or in the half-plane $v < 0$ can uniquely be written in the form $v = v(u)$, representing a solution of (2d). The origin $u = v = 0$ is a singular point of (2c), *i.e.* a point in which both of the right sides of (2c) vanish. Under the condition (5) there is no other singular point of (2c), $u \geqq 0$. The simplicity of the whole discussion is chiefly due to this circumstance. We mark the zero-lines, and the positive and negative regions of the right sides of (2c) (fig. 1). The zero-line $\dot{u} = 0$ coincides with the u-axis. The zero-line $\dot{v} = 0$ starts at the origin, wandering monotonically to $u \to +\infty$, $v \to -\infty$.

Lemma 1.—Each solution-curve $u(t)$, $v(t)$ of (2c) starting in a point of the positive v-axis goes monotonically falling to the right before crossing the u-axis vertically. After crossing, the curve goes monotonically falling to the left till it crosses horizontally the zero-line $\dot{v} = 0$. Then the curve monotonically rises, going further to the left, and either reaches a point of the negative v-axis for a finite value of t or tends to the origin for $t \to +\infty$.

This statement is almost obvious from fig. 1. Let $u(t_0) = 0$, $v(t_0) = v_0 > 0$. As long as $v > 0$, $u(t)$ increases and $v(t)$ decreases, t increasing. It follows that $v < v_0$ and $u(t) < v_0(t - t_0)$. With regard

to (5) the second equation (2c) implies $\dot{v} < -u^n$. Let t_1 be $> t_0$ but sufficiently near to t_0. We have $u(t_1) = u_1 > 0$, $v(t_1) < v_0$, and $\dot{v} < -u_1^n$, $t > t_1$; therefore $v(t) < v_0 - u_1^n(t - t_1)$ as long as $v > 0$. Thus the curve must cross the u-axis for some $t < t_1 + v_0 u_1^{-n}$. After crossing vertically u and v decrease, t increasing further. Evidently the curve cannot avoid the line $\dot{v} = 0$. After crossing, the curve goes further to the left, monotonically rising but necessarily remaining below that line. There are of course only two possibilities: the curve remains in the region $\dot{u} < 0$, $\dot{v} > 0$ either for all large values of t or only in a finite range $t < t$. In each case the curve tends necessarily to a limit point

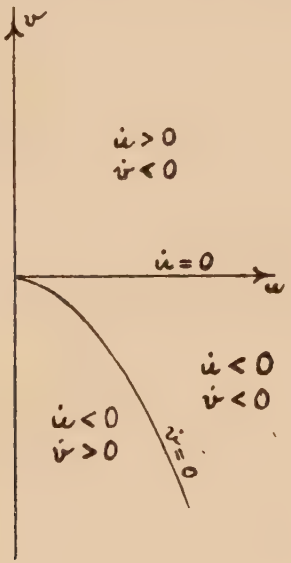

Fig. 1.

$u \geqq 0$, $v \leqq 0$. In the first case u, v must be a singular point of (2c), thus the origin. In the second case obviously $u^* = 0$, but $v^* < 0$, for otherwise we would have two different solutions of (2c) going through the same point (the origin), namely $u(t)$, $v(t)$ and the solution $u \equiv v \equiv 0$, in contradiction to the uniqueness-theorem.

Lemma 2.—Each solution of (2c) is a solution of the type described in Lemma 1.

We have to show that, t decreasing, each solution-curve must reach a point of the positive v-axis. We consider a solution starting in the region $\dot{u} = v < 0$, $\dot{v} > 0$ (this being the most unfavourable case) and let t decrease. As long as the curve remains in this region we have, on account of the second equation (2c),

$$v < -\frac{\mu(\mu - 1)u + u^n}{2\mu - 1} \quad . \quad . \quad . \quad (6)$$

and on account of (2*d*)

$$\frac{dv}{du} > -(2\mu - 1),$$

thus

$$v > v_1 - (2\mu - 1)(u - u_1), \quad . \quad . \quad . \quad (7)$$

u_1, v_1 being the starting-point. As (6) and (7) obviously contradict each other for large values of u, the solution-curve $v = v(u)$ must leave the mentioned region, u increasing. After leaving, $v \equiv v(u)$ increases again, u increasing. As long as $\bar{v} < v < 0$ we have (u sufficiently large)

$$\frac{dv}{du} > -(2\mu - 1) + \frac{\mu(\mu - 1)u + u^n}{|\bar{v}|} > \gamma u^n, \qquad (\gamma > 0)$$

thus $v > \frac{\gamma}{n+1} u^{n+1} - \beta$, and we conclude that the curve must cross the u-axis. After crossing (vertically) we can represent the solution again by the form $v(u)$, and now v increases, u decreasing. Equation (2*d*) shows that $\left|\frac{dv}{du}\right|$ is bounded, u decreasing, and we easily conclude that $v(u)$ can be continued to a point of the positive v-axis.

Now we prove the existence of an important solution-curve tending asymptotically to the origin. For this purpose we consider the solutions $v(u, v_0)$ of (2*d*) determined by the condition

$$v(0\,;\, v_0) \equiv v_0 < 0.$$

We show that all these solutions exist in the whole range $0 < u < (\frac{1}{4})^{\frac{1}{n-1}}$ and satisfy the inequality

$$v(u\,;\, v_0) < -\mu u + 2u^n, \qquad 0 < u < (\tfrac{1}{4})^{\frac{1}{n-1}}, \qquad v_0 < 0 \qquad (8)$$

With regard to the uniqueness-theorem we have

$$v(u\,;\, v_0) < v(u\,;\, v_0'). \qquad v_0 < v_0'.$$

(8) being proved, let v_0 increase and tend to zero. Obviously we obtain a limit function

$$v_E(u) = \lim_{v_0 \to -0} v(u\,;\, v_0)$$

satisfying, of course, the equation (2*d*) in the range $0 < u < (\frac{1}{4})^{\frac{1}{n-1}}$. We shall later show that this E-solution and the corresponding solution $u_E(t)$, $v_E(t)$ of (2*c*) leads to Emden's solutions of (2).

In order to prove (8) we may compare the solutions $v(u\,;\, v_0)$ with the solutions of the simpler equation ($\delta > 0$)

$$\frac{dw}{du} = -(2\mu - 1) - [\mu(\mu - 1) + \delta]\frac{u}{w} \quad . \quad . \quad (9)$$

the corresponding system of two equations being purely linear. The solution $w(u)$ of (9) satisfying the initial condition $w(0) = w_0 \gtrless 0$ is

easily seen to be determined by the transcendental equation

$$\left(\frac{w + c_1 u}{w_0}\right)^{c_1} = \left(\frac{w + c_2 u}{w_0}\right)^{c_2} \quad . \quad . \quad . \quad (10)$$

$-c_1$, $-c_2$ being the roots of the quadratic equation

$$x^2 + (2\mu - 1)x + \mu(\mu - 1) + \delta = 0;$$

therefore

$$c_1 = \mu - \tfrac{1}{2} + \sqrt{\tfrac{1}{4} - \delta}, \qquad c_2 = \mu - \tfrac{1}{2} - \sqrt{\tfrac{1}{4} - \delta} \quad . \quad (11)$$

For $\delta < \frac{1}{4}$ we have $\sqrt{1 - 4\delta} > 1 - 2\delta$, thus paying regard to (5)

$$c_1 > \mu - 2\delta > 0, \qquad 0 < \delta < \tfrac{1}{4} \quad . \quad . \quad . \quad (12)$$

Now let us consider the particular solution

$$w = -c_1 u \quad . \quad . \quad . \quad . \quad (10a)$$

of (9) obtained from (10) by $w_0 \to -0$. By (2*d*) we obtain

$$\frac{dv}{du} < -(2\mu - 1) - (\mu(\mu - 1) + \delta)\frac{u}{v}; \qquad u < \delta^{\frac{1}{n-1}}, \qquad v < 0,$$

and by subtracting this from (9)

$$\frac{d\Delta}{du} > \frac{\mu(\mu - 1) + \delta}{vw}\Delta; \qquad \Delta = w - v, \qquad u < \delta^{\frac{1}{n-1}}, \qquad v < 0 \quad (13)$$

More exactly, we have

$$\Delta = -c_1 u - v(u; v_0), \qquad v_0 < 0.$$

With regard to $\Delta(0) > 0$ we conclude from (13) that $\log \Delta$ increases, u increasing, as long as $v < 0$, $u < \delta^{\frac{1}{n-1}}$. *A fortiori* (see (12))

$$v(u; v_0) < -c_1 u < -(\mu - 2\delta)u \quad . \quad . \quad (14)$$

holds as long as $v < 0$, $u < \delta^{\frac{1}{n-1}}$, $\delta < \frac{1}{4}$, thus as long as $u < \delta^{\frac{1}{n-1}}$, $\delta < \frac{1}{4}$ without the restriction $v < 0$. Finally by fixing u and by making $\delta \to u^{n-1}$ we obtain from (14) at once the required inequality (8).

The E-solution $v_E(u)$, having now a definite significance, can be shown to satisfy the following inequalities:

$$v_E(u) \leqq -\mu u + 2u^n, \qquad 0 < u < (\tfrac{1}{4})^{\frac{1}{n-1}} \quad . \quad . \quad (15)$$

$$v_E(u) \geqq -\mu u, \qquad \dot{u} > 0 \quad . \quad . \quad . \quad . \quad . \quad (15a)$$

the latter of which remains to be proved. From (2*d*) we conclude

$$\frac{dv_E}{du} > -(2\mu - 1) - \mu(\mu - 1)\frac{u}{v_E},$$

as long as $v_E < 0$. By subtracting (9) from this inequality, choosing $\delta = 0$, we obtain

$$\frac{d\Delta}{du} > \frac{\mu(\mu - 1)}{wv_E}\Delta, \qquad \Delta = v_E - w \qquad . \qquad . \qquad (16)$$

By definition of the E-curve we have $v_E \to 0$, $u \to 0$, thus $\Delta(0) > 0$, if we consider a solution $w(u)$ of (9), $\delta = 0$, starting with a negative value $w(0) = w_0 < 0$. From (16) we obtain by a conclusion already applied $\Delta > 0$; thus $v_E > w$ in each u-interval, in which $v_E < 0$, $w < 0$. Now by letting w_0 tend to zero, thus letting w tend to the particular solution (10*a*), (11), $\delta = 0$, we get at once the inequality (15*a*). We conclude by Lemma 2 that the complete E-curve must lie entirely above the straight line $v = -\mu u$.

On account of (15) and (15*a*) the E-solution possesses the important property

$$\lim_{t \to +\infty} \frac{v(t)}{u(t)} = -\mu.$$

Now let us prove that there exists no other solution satisfying this relation. If v, v' are two different solutions of (2*d*), $v \sim -\mu u$, $v' \sim -\mu u$, we may assume $v < v'$, $u > 0$. Thus we have

$$D = v' - v > 0, \qquad \lim_{u \to 0} \frac{D}{u} = 0 \qquad . \qquad . \qquad (17)$$

From (2*d*) we obtain

$$\frac{dD}{du} = (\mu(\mu - 1)u + u^n)\frac{D}{vv'},$$

therefore, according to $vv' \sim \mu^2 u^2$ and to a well-known rule,

$$\lim_{u \to 0} \frac{\log D}{\log u} = \lim_{u \to 0} \frac{d \log D}{d \log u} = \lim_{u \to 0} \frac{u}{D}\frac{dD}{du} = 1 - \frac{1}{\mu},$$

thus

$$\lim_{u \to 0} \frac{\log \frac{D}{u}}{\log u} = -\frac{1}{\mu},$$

in contradiction to (17), leading to a non-negative limit on the left side. The whole argument is independent of the restriction (5), but for the proof of the properties (15) of the E-curve the condition (5) is an essential one.

Arrangement of the solutions of (2*c*) *in the u-v-plane.*—Let us trace the whole E-solution in the u-v-plane, $u \geq 0$. According to Lemma 2 it must start at a point $u = 0$, $v = v_E{}^0$ of the positive v-axis. A solution starting at a point $u = 0$, $v_0 > 0$, and $v_0 < v_E{}^0$, called an M-solution by Fowler, must remain entirely inside the region bounded by the E-curve and the part $0 < v < v_E{}^0$ of the v-axis, according to Lemma 1, necessarily tending to the origin, $t \to +\infty$. However a solution-curve starting at a point $u = 0$, $v_0 > v_E{}^0$—we call it an F-solution—must remain outside the mentioned region and reach a point $u = 0$, $v_1 < 0$ of

the negative v-axis for a finite value of t according to the definition of the E-curve and to Lemma 1. Therefore the whole family of the solution-curves of (2c), $u \geqq 0$, is divided in two regions by the E-curve, the region of the M-solutions and the region of the F-solutions (fig. 2).

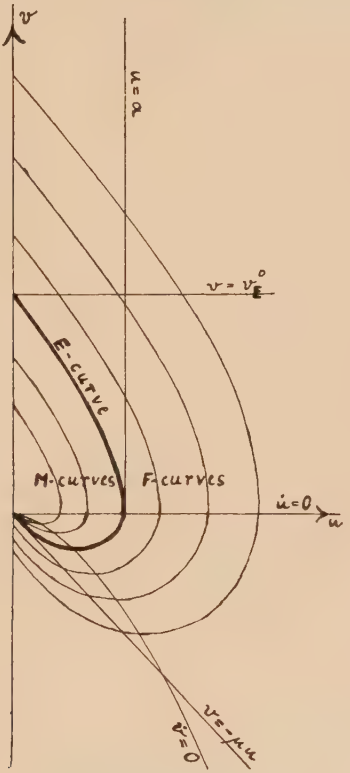

FIG. 2.—Typical behaviour of the solutions of (2c), $\mu \geqq 1$.

§ 3. *Relative Arrangement and Asymptotic Behaviour of the Solutions of* (2), $\xi \to 0$.—According to our chief aim let us now derive the astrophysically important properties of the solutions $\theta(\xi)$ of (2) by going back from t, u, v to the original variables ξ, θ. To a definite solution $u(t)$, $v(t)$ of (2c) corresponds by (4) a definite solution $\theta(\xi)$ of (2) and reverse. The solution $u(t + c)$, $v(t + c)$, defining the same curve in the u-v-plane, leads to the solution $A^{\mu}\theta(A\xi)(A = e^{-c})$ of (2). Therefore we have a one-to-one correspondence between the solution-curves of (2c) and the "homologous" groups of solutions of (2). According to this relation let us keep the concepts of E-, M-, and F-solutions in case of the equation (2).

By (4) we easily obtain the formulæ

$$u = \xi^{\mu}\theta, \quad v + \mu u = \dot{u} + \mu u = -\xi^{\mu+1}\frac{d\theta}{d\xi} \qquad . \quad (18)$$

At first we have to justify the name "E-solution" by proving the existence of $\lim_{\xi \to 0} \theta(\xi)$, the well-known property of Emden's solutions.

From (15) and (15a) we obtain $\frac{\dot{u}}{u} \to -\mu$, $t \to +\infty$, thus $u = e^{-(\mu+\epsilon)t}$; $\epsilon \to 0$, $t \to +\infty$. The inequalities (15), (15a) can be written in the form

$$0 \leqq \frac{d\theta}{dt} = \frac{d(e^{\mu t}u)}{dt} \leqq 2e^{\mu t}u^n.$$

Hence follows with regard to (3), $(n-1)\mu = \lambda$,

$$0 \leqq \frac{d\theta}{dt} \leqq 2e^{-(\lambda+\epsilon)t}; \quad \epsilon \to 0, \; t \to +\infty,$$

from which the existence of $\lim_{\xi \to 0} \theta > 0$ is evident ($\lambda > 0$). For the sake of completeness we prove that these solutions are the only solutions possessing Emden's property, $\theta \to \theta_0 > 0$, $\xi \to 0$. According to (18) and to a well-known rule we have

$$\lim_{t \to \infty} \frac{v}{u} + \mu = -\lim_{\xi \to 0} \frac{\xi}{\theta}\frac{d\theta}{d\xi} = -\frac{1}{\theta_0}\lim_{\xi \to 0} \frac{\xi^2 \frac{d\theta}{d\xi}}{\xi} = -\frac{1}{\theta_0}\lim_{\xi \to 0} \frac{d}{d\xi}\left(\xi^2 \frac{d\theta}{d\xi}\right) = 0,$$

leading necessarily to the E-curve constructed in § 2.

Each F-solution $\theta(\xi)$ must cease to be positive, ξ decreasing, and have a zero before ξ reaches zero according to the statements of § 2.

Each M-solution can be continued to arbitrarily small values of ξ. Let us consider now the asymptotic behaviour near $\xi = 0$. For this purpose we have to distinguish between the entirely different cases $\mu > 1$ and $\mu = 1$.

$\mu > 1$. It is convenient to consider θ as function of $x = \frac{1}{\xi}$, satisfying the equation (2a) and being positive for all large values of x. According to (2a) we have $\frac{d^2\theta}{dx^2} < 0$; thus $\theta = \theta\left(\frac{1}{x}\right)$ represents a convex curve lying entirely above the x-axis. Hence $\frac{d\theta}{dx}$ must be positive and decrease, x increasing, thus

$$\lim_{x \to \infty} \frac{d\theta}{dx} = \lim_{x \to \infty} \frac{\theta}{x} = \lim_{\xi \to 0} \xi\theta = C \geqq 0 \quad . \quad . \quad (19)$$

exists. We show that for the M-solutions C must be positive by proving that $C = 0$ necessarily leads to E-solutions. For in the case $C = 0$ let α be the smallest number, so that

$$\frac{d\theta}{dx} = o(x^{\alpha+\delta}), \quad \delta > 0 \quad . \quad . \quad . \quad (20)$$

for each $\delta > 0$. According to $C = 0$ and (19), we must have

$$\alpha \leqq 0 \quad . \quad . \quad . \quad . \quad . \quad (21)$$

If we are able to prove $\alpha < -1$, we can immediately conclude from (20) that θ remains bounded, tending necessarily to a positive limit according to the convexity of the curve $\theta = \theta\left(\frac{1}{x}\right)$. The assumption $\alpha \geqq -1$ leads to the following contradiction. By (20) we obtain

$$\theta = o(x^{\alpha+1+\delta}), \quad \delta > 0,$$

thus by (2a)

$$\frac{d^2\theta}{dx^2} = o(x^{n(\alpha+1)-\lambda-2+n\delta}), \quad \alpha + 1 \geqq 0.$$

By integrating this equation from x to $+\infty$, and taking account of $\lim_{x\to\infty} \frac{d\theta}{dx} = 0$, we get

$$\frac{d\theta}{dx} = o(x^{n(\alpha+1)-\lambda-1+n\delta}),$$

$\delta > 0$ and arbitrarily small. By definition of the number α we must have $n(\alpha+1) - \lambda - 1 + n\delta \geqq \alpha$, thus, δ being arbitrarily small, $n(\alpha+1) - \lambda - 1 \geqq \alpha$ or $\alpha \geqq \frac{\lambda}{n-1} - 1 = \mu - 1 > 0$ in contradiction to (21). Thus we get for M-solutions

$$\theta \sim \frac{C}{\xi}, \quad C > 0 \quad . \quad . \quad . \quad . \quad (22)$$

$\mu = 1$. This case must be treated by other methods. Let us go back to the equation (2c), *i.e.* to the equation

$$\frac{dv}{du} + 1 + \frac{u^n}{v} = 0 \quad . \quad . \quad . \quad . \quad (23)$$

and apply a very useful theorem of Hardy used already by Fowler.* First, however, let us remark, according to the results of § 2, that each M-solution must satisfy the inequalities

$$-u < v < 0. \quad u > 0.$$

$-\frac{dv}{du} \succ 1$ being thus impossible, and $-\frac{dv}{du} \sim 1$ likewise, (leading to the E-curve), by Hardy's theorem there only remains $-\frac{dv}{du} \prec 1$. Therefore by (23)

$$v = \dot{u} \sim -u^n.$$

This leads to

$$u \sim \left[\frac{1}{(n-1)(t-c)}\right]^{\frac{1}{n-1}},$$

* Fowler, *loc. cit.*

and thus ($C = e^{-c}$) by (4) to

$$\theta \sim \frac{1}{\xi}\left[\frac{1}{(n-1)\log(C/\xi)}\right]^{\frac{1}{n-1}}. \quad . \quad . \quad . \quad (24)$$

Now we are able to prove Fowler's chief theorem about the solutions of (2). To a given set of initial values

$$\xi = \xi_0, \quad \theta = \theta_0 \geqq 0, \quad \left(\frac{d\theta}{d\xi}\right)_{\xi=\xi_0} = \theta_0' \quad . \quad . \quad (25)$$

corresponds by (18) a definite point $u_0 \geqq 0$, v_0 of the u-v-plane,

$$u_0 = \xi_0^{\mu}\theta_0, \qquad v_0 + \mu u_0 = -\xi_0^{\mu+1}\theta_0'. \quad . \quad . \quad . \quad (26)$$

Let us fix a certain point ξ_0, θ_0 of the ξ-θ-plane, and let θ_0' pass through all starting slopes. Then the corresponding point u_0, v_0 in the u-v-plane describes a vertical straight line $u_0 = \text{const.}$ Let $u = \alpha$, $v = 0$, be the point where the E-curve crosses the *u-axis*. By means of the statements of § 2 (see fig. 2) we obviously obtain Fowler's theorem, viz.—

Theorem.—Starting-points ξ_0, θ_0 are divided into two classes by a critical curve $\theta = \alpha\xi^{-\mu}$, α being calculable in terms of n and μ.

Any solution $\theta(\xi)$ starting in a point ξ_0, θ_0 above the critical curve is an F-solution with a zero at some positive $\xi < \xi_0$.

If the starting-point lies on the critical curve, all solutions are again F-solutions except the case where the starting slope touches the critical curve, this case leading to an E-solution.

If the point ξ_0, θ_0 lies below the critical curve, two starting slopes $\theta_0' = \omega_1$, $\theta_0' = w_2$, $\omega_2 < \omega_1 < 0$ exist, leading to E-solutions. All solutions corresponding to $\theta_0' > \omega_1$ or $\theta_0' < \omega_2$ are F-solutions. Slopes between ω_1 and ω_2, however, lead to M-solutions, becoming infinite at $\xi = 0$ and being of the asymptotic form (22) for $\mu > 1$, of the form (24) for $\mu = 1$. In the case $\theta_0 = 0$, ω_1 becomes zero, and $\theta_0' = \omega_1 = 0$ yields the trivial solution $\theta \equiv 0$.

Each solution starting at a point ξ_0, $\theta_0 > 0$ must have a zero at some $\xi > \xi_0$. The E- and M-solutions are monotonic functions in their whole positive range. An F-solution, however, possesses a single maximum between its two zeros.

The latter statements are obvious from (2*a*).

§ 4. *Real Continuation of the Solutions of* (2) *to Negative Values of* θ.—A solution of (2) is of course only of astrophysical interest as far as $\theta \geqq 0$. For the sake of completeness, however, let us consider the possibility of a real continuation to $\theta < 0$, and regain Fowler's results in the non-trivial case

$$n = \frac{p}{q} > 1, \qquad \mu \geqq 1; \qquad p, q \text{ odd},$$

containing the case of Emden's equation $n = 3$. Considering the equations (2*c*) we have in this case

$$(-u)^n = -u^n.$$

Thus, if $u(t)$, $v(t)$ satisfy (2*c*), $-u(t)$, $-v(t)$ is also a solution of (2*c*).

Consider now the point $u = 0$, $v = v_E^0$, where the E-curve $u_E(t)$, $v_E(t)$ cuts the positive v-axis, and mark the piece

$$u = 0, \qquad -v_E^0 \leqq v \leqq v_E^0 \qquad . \qquad . \qquad . \qquad (27)$$

of the v-axis. An arbitrarily given solution-curve ($u \geqq 0$) of (2c) must cross the negative v-axis in a point $v_0 < 0$ and pass to $u < 0$, t increasing. Application of Lemma 1 to $-u(t)$, $-v(t)$ shows that the curve must cross the positive v-axis in a point $v_1 > 0$, and pass to $u > 0$ again. In such a way we clearly obtain a sequence of successive crossing-points

$$u = 0, \qquad v_0 < 0, \qquad v_1 > 0, \qquad v_2 < 0, \qquad v_3 > 0 \ . \ . \ .$$

on the v-axis. Let $t_0 < t_1 < t_2 < t_3 < \ . \ . \ .$ the sequence of the corresponding crossing-times. By integration of the equation

$$\frac{d}{dt}\left(\frac{v^2}{2} + \frac{\mu(\mu-1)}{2}u^2 + \frac{u^{n+1}}{n+1}\right) = -(2\mu-1)v^2, \qquad . \qquad (28)$$

being identical with (2c), along $t_\nu < t < t_{\nu+1}$, we get

$$v_{\nu+1}^2 - v_\nu^2 = -(2\mu-1)\int_{t_\nu}^{t_{\nu+1}} v^2 dt > 0;$$

thus

$$-v_0 > v_1 > -v_2 > v_3 > -v_4 > \ . \ . \ .$$

Now these v_ν cannot lie entirely outside the range (27), else we would have $v_{2\nu+1} \to v^*$, $v_{2\nu} \to -v^*$, and the curve starting in $u = 0$, $v = v^*$ would cross the v-axis in the point $u=0$, $v=-v^*$, in contradiction to (28). Thus there is a final crossing in a point $u = 0$, $v = v_i$ outside (27), $|v_i| > v_E^0$. According to the statements at the end of § 2 applied to $(-1)^{i+1}u(t)$, $(-1)^{i+1}v(t)$, we must have a further crossing $u = 0$, $v = v_{i+1}$, but $|v_{i+1}| \leqq v_E^0$. This crossing now must be the last one of all, for after crossing the curve $(-1)^i u(t)$, $(-1)^i v(t)$ must be an M-curve or an E-curve tending asymptotically to the origin.

Returning to the equation (2) we easily conclude:

Any solution $\theta(\xi)$ of (2) cuts the ξ-axis at most a finite number of times, ξ decreasing, $\pm\theta(\xi)$ finally being an E-solution or an M-solution with the asymptotic form (22) for $\mu > 1$ and of the form (27) for $\mu = 1$ near $\xi = 0$.

Algol and Stellar Rotation. By O. Struve and C. T. Elvey.

1. It is a well-known fact that the spectra of many early-type stars show extremely wide and hazy absorption lines. In a number of papers we have shown that these "dish-shaped lines" may be interpreted as being due to axial rotation. The evidence thus far available may be summarized as follows:—

(1) The theoretical contours computed on the basis of axial

rotation* are in agreement with the observed contours of dish-shaped lines.†

(2) The values derived for the components, in the line of sight, of the equatorial velocities of rotation for single stars, range from zero to about 250 km./sec. Assuming that the axes of the stars having the largest rotational velocities are at right angles to the line of sight, it appears that these stars are stable, though some of them may not be far removed from the stage where instability sets in.‡

(3) The broadening ascribed to rotation affects all lines of all elements (excepting those due to interstellar gases), and the amount of broadening is in agreement with the requirements of the Doppler effect.‡

(4) In spectroscopic binaries of short period (in which we have reason to believe that the periods of rotation and of revolution coincide), we find that dish-shaped lines are observed in every case where the period is very short and at the same time the amplitude is large. This agrees with the requirements of the theory.

(5) We may subdivide all known causes of broadening of absorption lines into two main classes: those of the first class depend upon the physical properties of the individual atoms or of the absorbing gases, while those of the second class are of a mechanical nature, and are directly related to Doppler effect. Of the first set of causes, the most important is the one depending upon the abundance of the atoms in the absorbing atmosphere. This "abundance broadening" has been exhaustively treated by a number of investigators, and it is clear that it cannot be responsible for the uniformly broad and shallow lines of all elements. It is possible to show that the observed shape of the lines cannot be due to an unusual form of the relation between absorption coefficient and wave-length. We have treated this point briefly in our discussion of the multiplet intensities of *Si III* in stellar spectra.§ Consider, for example, the broad lines λ 4553, λ 4568, and λ 4575 in ϵ and κ Orionis. The resolving power of the three-prism Bruce spectrograph, used for making the spectrograms, is sufficient to guarantee the reality of the contours. Let us suppose for a moment that the shape of the lines is due to the form of the absorption coefficient. Then we derive the value of $\sigma_1 l$ from the equation $I = I_0 \frac{1}{1+\sigma_1 l}$, which, according to Unsöld, represents the contour of a line produced by a scattering atmosphere. It should be noted that the exact form of the equation is not essential: we could use equally well the exponential formula $I = I_0 e^{-\sigma_1 l}$.

In the silicon triplet under discussion ($2^3S - 3^3P$) the theoretical ratio of the intensity of the strongest line to the weakest is 5 : 1. This, of course, means that five times as many atoms contribute to the strongest line λ 4553 than contribute to the weakest line λ 4575. The absorption coefficient is proportional to the number of atoms; consequently, if

* Shajn and Struve, *M.N.R.A.S.*, **89**, 222, 1929.
† Elvey, *Ap. J.*, **71**, 221, 1930.
‡ *Ibid.*, **72**, 1, 1930. § *Ibid.*, **72**, 267, 1930.

the absorption coefficient for the strongest line is known, that for the weakest line is given by $\sigma_2 = \frac{\sigma_1}{5}$. The theoretical contour of the line λ 4575 may now be constructed by substituting σ_2 into the initial equation for I. The result disagrees completely with the observed contour: the computed contour of the line λ 4575 covers a much smaller area than does the observed contour. If we take the ratio of the areas of the observed lines we find a value close to $\sqrt{5}$. The ratio of the theoretical contours is approximately $\gtrsim 5$. Consequently we may state that the observed contours of the broad lines cannot be due to the form of the absorption coefficient. This applies to such broadening agencies as turbulence, thermal agitation, Stark effect, etc., for all of them would require the observed intensities to be proportional to N, instead of to $\sqrt{N}$.

There remain two plausible explanations, viz.: (*a*) rotation, and (*b*) convection currents on a large scale which involve deep layers of the stellar atmosphere. The evidence mentioned under (4), derived from the spectroscopic binaries, is definitely in favour of rotation, for there is no reason to suppose that convection currents would be more powerful in close binaries than in distant ones.

2. The study of eclipsing variables furnishes an independent method of determining the rotational velocities. In these stars, axial rotation causes the spectrum lines to be unsymmetrical at the time of partial eclipse, when one limb of the brighter star is covered, while the other limb is not. This effect has been investigated by Schlesinger, Rossiter, McLaughlin, and J. S. Plaskett, and their results have shown conclusively that appreciable rotational velocities are present in a number of stars. We shall attempt to show that the method adopted by us for determining rotational velocities gives values identical with those obtained from the "limb-effect."

We have concentrated our attention on Algol, since this is the only eclipsing variable bright enough for our three-prism spectrograph. The earlier investigation of McLaughlin * was carried out with an instrument of smaller dispersion, and the contours of the lines on his plates were probably largely determined by the resolving power of his equipment. The linear dispersion of our spectrograph was 10 A. per mm. at λ 4500. Fig. 1 shows the contour of the line *Mg* λ 4481 obtained from seven spectrograms of Algol when out of eclipse. The line is moderately flat, and is appreciably broadened. The contours of other lines in Algol agree with that of the magnesium line, except that the total intensities are smaller. We have, therefore, based our computations upon the magnesium line alone. The total area covered by λ 4481 in Algol corresponds very nearly to the total area of the same line in Vega, where rotational broadening cannot be detected with our dispersion. We have therefore adopted the line in Vega as a standard, and have computed the widened contour for several values of the limb-velocity, following the method we used in earlier work. The

* *Ap. J.*, **60**, 22, 1924.

continuous curve (fig. 1) represents the theoretical contour for an equatorial limb-velocity of one angstrom or 67 km./sec. It will be seen that the curve very nearly fits the observations : the observations have a slight tendency to fall below the curve in the centre, and above at the edges. We therefore adopt, as final value for the rotational velocity of Algol, V = 60 km./sec. It is difficult to make an estimate of the uncertainty attached to this value, but we believe from previous experience that it should be within ± 15 km./sec.

Our determination may now be compared with that obtained from the period of the binary. Adopting the size of the brighter star as

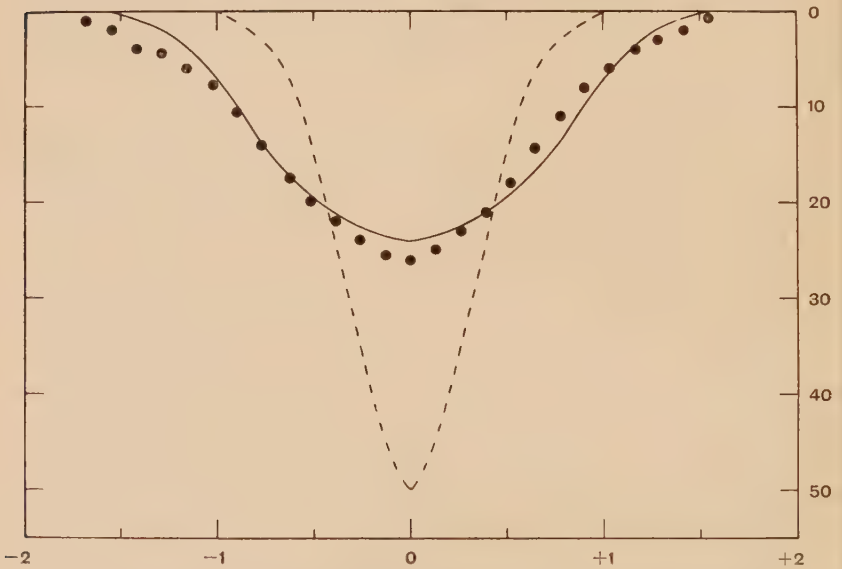

Fig. 1.—Contour of *Mg* λ 4481 in Algol when out of eclipse. Abscissæ are angstroms and ordinates are percentages of absorption. The observations are shown by dots. The broken curve represents the assumed contour of the line for a non-rotating star, and the continuous curve represents the computed contour for a rotational velocity of one angstrom or 67 km./sec.

given by McLaughlin, we find the computed velocity to be 55 km./sec. The agreement is excellent, and leaves no doubt of the correctness of our procedure.

Since the rotational velocity is known, we may predict the contour at the time of partial eclipse. Fig. 2 shows the theoretical contours of the line λ 4481 for a number of phases preceding central eclipse. We have assumed that the axis of the star is perpendicular to the plane of the orbit, and that the equatorial rotational velocity amounts to one angstrom. The relative sizes of the two stars and the elements of the orbit were taken from the work of McLaughlin. It was also assumed that the eclipsing star is completely dark. This is not quite correct, but it is difficult to make allowance for the additional light from the com-

panion, for its spectral type is not known and we cannot even predict with any certainty whether an absorption line or a portion of the continuous spectrum is superposed on the spectrum of the brighter

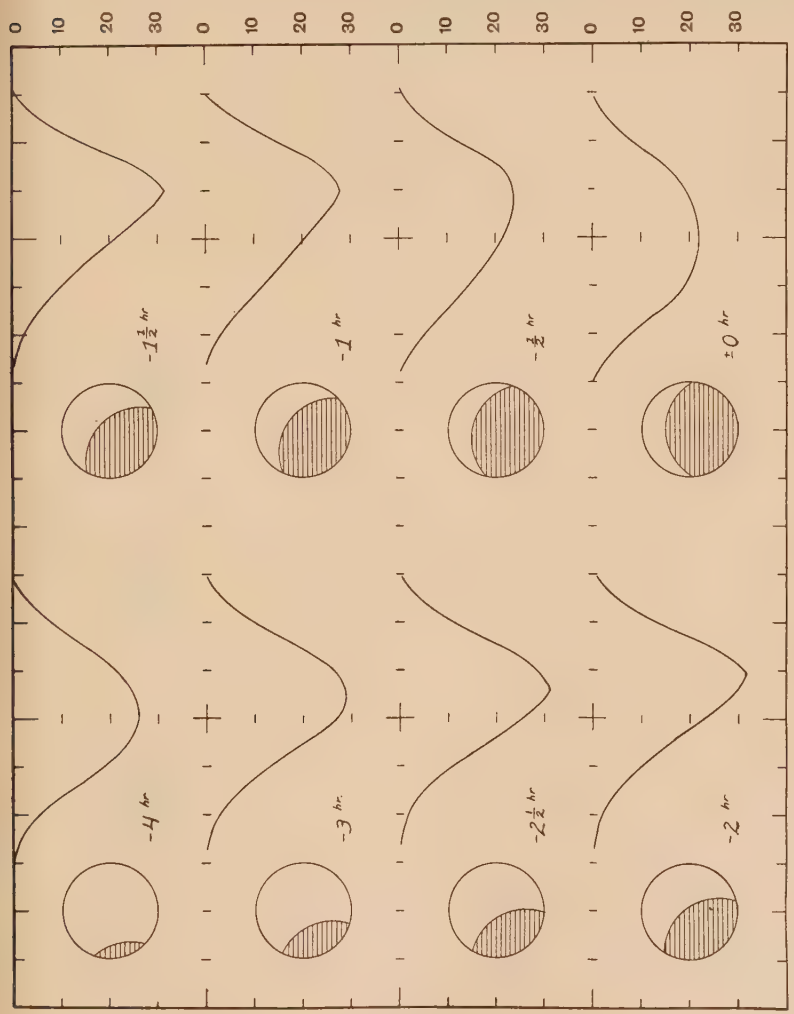

FIG. 2.—Computed contours of *Mg* λ 4481 in Algol during eclipse. The phases are counted from central eclipse. The abscissæ are angstroms and the ordinates are percentages of absorption.

star. It is of interest to note that there should be an appreciable amount of asymmetry near phase ± $0^d.05$. At about the same phase the depth of the lines is slightly greater than when the star is out of eclipse, and this probably accounts in part for the increased intensity of the extra lines observed during eclipse by Miss Ida Barney.* Exactly at central eclipse the line is wider than normal, and slightly flatter. Our

* *Ap. J.*, **35**, 95, 1923.

computations were carried out for circular discs. If the stars have any appreciable flattening, the curves will be slightly altered and the abnormal features will be enhanced.

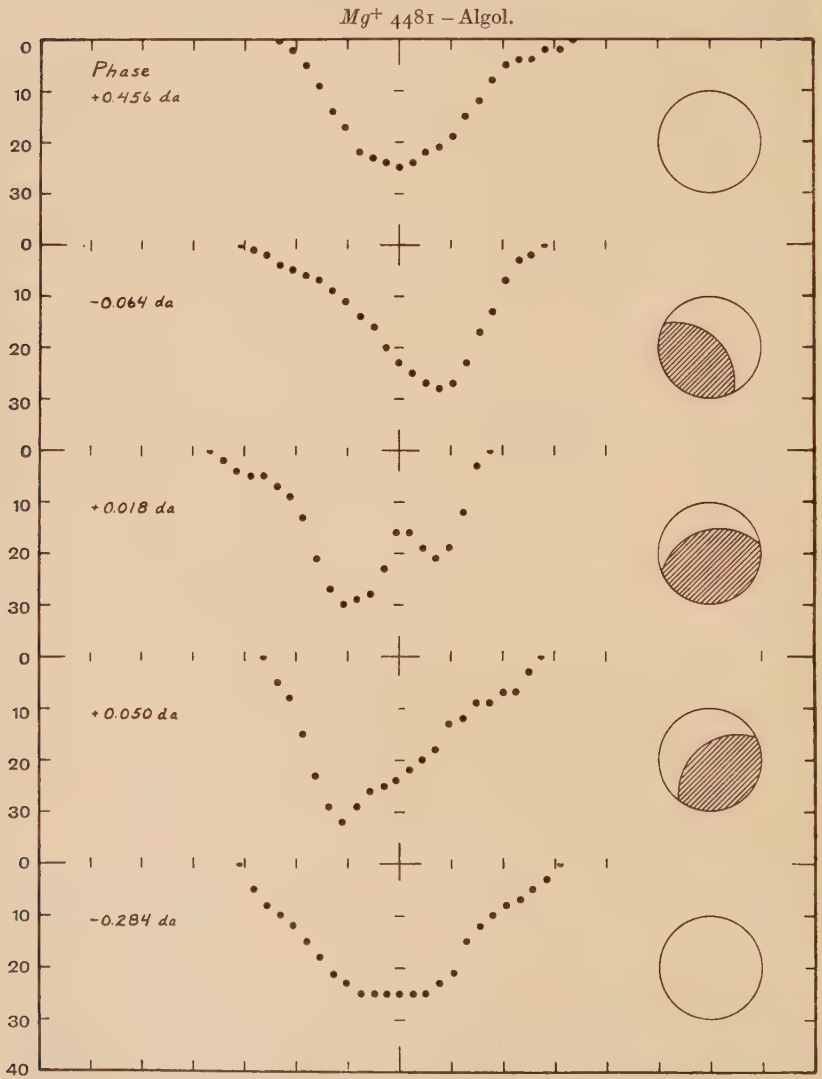

Fig. 3.—Observed contours of *Mg* λ 4481 in Algol. Coordinates as in fig. 2.

Our dispersion proved to be just sufficient to detect the asymmetry of the lines in Algol during eclipse. Fig. 3 shows a few selected contours on which the asymmetry before and after central phase is well

shown. The first and last contours were taken when the star was out of eclipse, while the middle contour falls near central phase. In the last-named there is a pronounced doubling of the line, which we at first believed to be real. However, later plates have failed to substantiate this duplicity, and we are now inclined to regard it as accidental. The amount of asymmetry shown in the observed contours agrees well with the computed values.

In fig. 4 we have plotted the ratio of the area towards the violet from the deepest point, to the area towards the red. These values cluster around unity outside of eclipse (since at that time the line is symmetrical). The theoretical curve of the ratio, computed for our velocity of one angstrom, agrees well with the observations. The

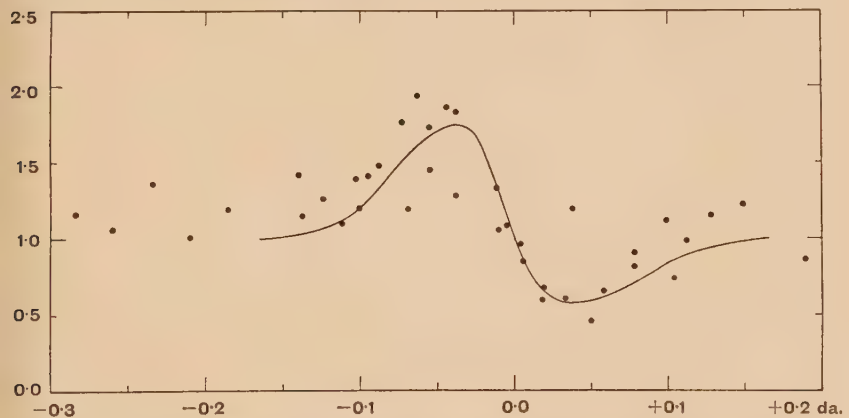

Fig. 4.—Ratio of area towards the violet from deepest point of *Mg* λ 4481 to area towards the red. Abscissæ are phases in days.

scattering, while large, is fully accounted for by the uncertainty of the measurements.

Table I. gives the radial velocities as determined from our plates. We have aimed in all cases to measure the deepest point of each line. In order to isolate the "limb-effect" we have subtracted from each measured value the component of the velocity due to the motion of the star in its orbits of short and long period. In this computation we have used the elements derived by McLaughlin. The corrected velocities, from plates marked "good" in Table I., are plotted in fig. 5.

The theoretical curve, also shown in the diagram, agrees well with the observations, and, of course, resembles the curve obtained by McLaughlin. However, our observations show an appreciably greater amplitude than his, due, no doubt, to the fact that our dispersion was greater. On the other hand, our amplitude is slightly less than that of the theoretical curve. This we attribute to the fact that even with our dispersion of 10 A. per mm. at λ 4500, the deepest point of the line was slightly obliterated by the finite resolving power of the instrument,

Table I.

Radial Velocities of Algol.

Heliocentric. Date U.T.	Heliocentric Phase.	Radial Velocity.	Quality (g=good; f=fair; p=poor).
1930.			
Nov. 5·176	+1·455	+13·9	g
11·990	−0·333	+32·7	p
12·015	− ·308	+45·0	g
12·038	− ·285	+46·2	g
12·062	− ·261	+39·8	g
12·088	− ·235	+29·8	g
12·112	− ·211	+34·7	g
12·136	− ·187	+34·6	g
12·160	− ·163	+31·2	g
12·185	− ·138	+31·2	g
12·219	− ·104	+37·3	g
12·267	− ·056	+41·2	g
12·360	+ ·037	−14·5	f
12·407	+ ·084	− 9·8	p
17·969	− ·088	+40·0	g
17·994	− ·063	+44·6	g
18·019	− ·038	+30·1	g
18·047	− ·010	+15·1	g
18·076	+ ·019	− 6·6	g
18·107	+ ·050	−20·6	g
18·135	+ ·078	−15·9	g
18·161	+ ·104	−11·0	g
18·185	+ ·128	−14·4	g
18·206	+ ·149	− 6·0	f
18·225	+ ·168	− 9·6	g
18·247	+ ·190	− 3·7	g
18·264	+ ·207	−10·4	f
18·278	+ ·221	−10·7	g
18·292	+ ·235	− 6·1	g
18·476	+ ·419	−12·7	p
23·034	+2·109	+52·5	g
23·442	−0·350	+42·1	g
Dec. 17·040	+0·310	−16·2	g
25·194	− ·138	+30·6	g
25·208	− ·124	+35·1	g
25·231	− ·101	+35·4	g

TABLE I.—*continued.*

Heliocentric. Date U.T.	Heliocentric Phase.	Radial Velocity.	Quality (g=good; f=fair; p=poor).
1930.			
Dec. 25·263	− ·069	+37·3	g
25·288	− ·044	+38·5	p
25·336	+ ·004	− 9·1	f
28·060	− ·140	+39·3	g
28·078	− ·122	+29·3	g
28·105	− ·095	+39·7	g
28·127	− ·073	+36·9	g
28·145	− ·055	+35·5	g
28·162	− ·038	+18·5	f
28·178	− ·022	+21·9	f
28·206	+ ·006	+ 6·8	g
28·233	+ ·033	−10·0	g
28·258	+ ·058	−21·0	g
28·278	+ ·078	−23·3	g
28·298	+ ·098	−20·1	g
28·312	+ ·112	−22·0	g
31·062	− ·005	+25·3	f
31·085	+ ·018	− 8·1	f

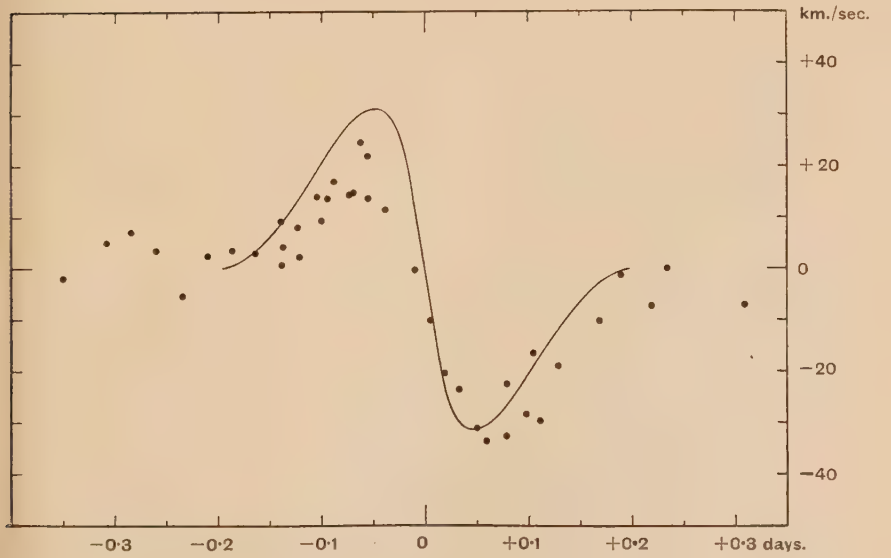

FIG. 5.—Rotational effect in the radial velocities of Algol.

and consequently the true asymmetry must have been greater than that recorded by the measurer.

There is also a slight displacement of the whole curve which can

perhaps best be attributed to a small error in the short-period elements used in deriving the corrections to the velocities. Unfortunately, our material does not permit us to derive a new set of elements for the epoch of our observations. If the orbit has changed a little since the time of McLaughlin's observations, this would fully account for the displacement. It will be seen that the observed curve passes through zero at zero phase, as it should according to theory. All phases used in this paper were computed from the epoch : heliocentric minimum J.D. 2426269·8838 + 2·8673218 E (G.M.T.), for which we are indebted to Professor Joel Stebbins.

An interesting feature of the contours remains to be pointed out : there is a tendency for the total absorbed energy in the line to vary with the phase. This is illustrated in Table II. The rotational

Table II.

Limits of Phase in Days.	No. of Plates.	Total Intensity in Units of 1A × 100 per cent. Absorption.
−·758 to −·186	10	·383
−·140 to −·055	13	·397
−·044 to +·056	16	·500
+·058 to +·235	10	·397

broadening of the line does not account for this change. In fact, our computations are made under the assumption that the total area of the line remains unaffected by rotational broadening. But this can be true only if the contour of the line is the same for all parts of the star's disc. We cannot at this time attribute the observed variation to any definite cause. It is possible that it is entirely instrumental—due perhaps to a tendency to under-expose the plates at the time of minimum.

3. There are several other eclipsing variables in which the conditions for detecting the effect of rotation are even more favourable than in Algol. Fig. 6 shows the predicted contours at central phase for 1 Hev. Cas. and TX Cas. Since in these stars the eclipses are nearly central, the absorption lines should display a tendency towards doubling. The curves were computed for a rotational velocity of 1 angstrom, or 67 km./sec. The linear dimensions of these stars are not accurately known, and it is not possible to predict the velocity of rotation. The drawings, therefore, illustrate merely the general character of the line. If the rotation should be greater than 67 km./sec., the doubling would be still more pronounced.

4. In a recent paper E. A. Kreiken * has discussed the rotational velocities determined by Elvey for fifty-nine stars, and has drawn from them the conclusion that the angular velocity is correlated with the linear diameter of the star. Kreiken's formula is :

$$\log R = 3{\cdot}34 - 1{\cdot}49 \log D,$$

D designates the hypothetical semi-diameter of the star, in units of

* *M.N.R.A.S.*, **91**, 251, 1930 ; *Proc. R. Acad. Amsterdam*, **33**, 1028, 1930.

0″·0001, as seen from a distance of 10 parsecs, and R is the "reduced velocity of rotation," in km./sec., at a distance $D = 10$ from the centre. Kreiken finds that a star for which $D = 10$ has a real semi-diameter of $16{\cdot}8 \times 10^5$ km. Consequently we may transform his equation to give us a relation between the observed velocity V and the linear semi-diameter d expressed in terms of the sun's radius. The resulting formula is :

$$\log V = 2{\cdot}04 - 0{\cdot}49 \log d.$$

For a star of the size of the sun, the rotational velocity would be 110 km./sec. Kreiken's curve depends upon material covering a range in D from 6 to 100 units, or, in terms of the sun's radius, from $d = 1{\cdot}4$ to $d = 24$. The formula takes no account of spectral type, and should accordingly hold for all stars indiscriminately.

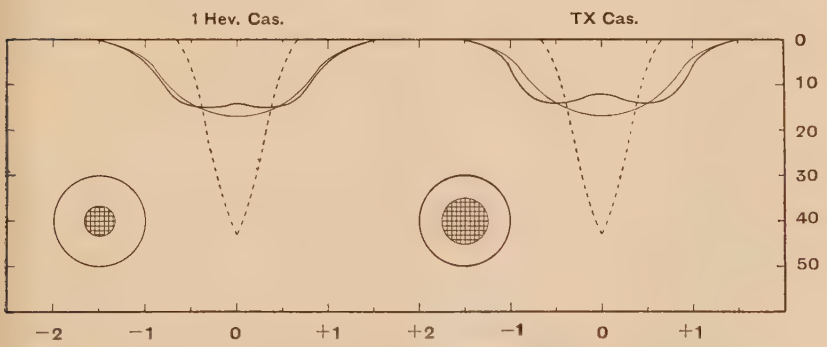

Fig. 6.—Computed contours of lines during central phase in 1 Hev. Cas. and in TX Cas. The broken curve represents the assumed contour of the line for a non-rotating star. The thin continuous curve is the contour when the star is out of eclipse, and the heavy line represents the contour at mid-eclipse.

It can be shown that this is not the case. If we apply the formula to giants and dwarfs of spectral types F0 to K we find many for which the velocity of rotation should exceed 50 km./sec., and should thus be easily recognizable on our spectrograms. A list of stars having diameters not exceeding four times that of the sun is given in Table III. The rotational velocities were estimated by comparing these stars with others for which the velocities had formerly been determined by Elvey. There is a strong correlation between spectral type and rotational velocity : all stars having large velocities are of type earlier than F5. The results are obviously unfavourable to the formula of Kreiken, especially for the later spectral classes.

A survey of stars having diameters larger than four times that of the sun supports the above conclusions : there are several stars that have appreciable rotational velocities among spectral classes earlier than F5, but there are none among the later types.

Evidently Kreiken's relationship holds only for stars of types earlier than F5, and it is probable that it merely reflects the criterion used by

Table III.

Star.	Mg.	Sp.	Par.	R.	Rotational Velocity in km./sec.
11 β Cas.	2·42	F5	·071	4	75
24 η Cas.	3·64	F8	·180	1	0
2 α Tri.	3·58	F5	·044	4	60
1 π^3 Ori.	3·31	F8	·124	2	0
55 δ Gem.	3·51	Fo	·056	3	70
10 α CMi.	0·48	F5	·310	2	0
16 ζ Cnc. A.	5·56	Go	·046	3	0
10 UMa.	4·09	F5	·070	2	25
25 θ UMa.	3·26	F8*p*	·060	4	0
29 υ UMa.	3·89	Fo	·038	4	70
53 ξ UMa.	4·41	Go	·140	1	0
78 ι Leo	4·03	F5	·057	3	0
5 β Vir.	3·80	F8	·098	2	0
29 γ Vir.	3·65	Fo	·085	3	25
42 Com.	5·22	F5	·058	2	20
4 τ Boo.	4·51	F5	·063	2	0
37 ξ Boo.	4·64	G5	·155	1	0
44 *i* Boo. br.	var.	Go	·078	2	0
51 μ Boo.	4·47	Fo	·032	3	50
41 γ Ser.	3·86	F5	·082	2	0
40 ζ Her.	3·00	Go	·106	3	0
86 μ Her.	3·48	G5	·112	2	10
70 Oph.	4·07	Ko	·198	1	0
30 δ Aql.	3·44	Fo	·057	3	100
13 θ Cyg.	4·64	F5	·067	1	0
60 β Aql.	3·90	Ko	·080	4	0
6 Hev. Cep.	4·63	Go	·050	4	0
65 τ Cyg.	3·82	Fo	·051	3	100
78 μ Cyg.	4·73	F5	·052	3	0
7 And.	4·62	Fo	·044	2	40
17 ι Pis.	4·28	F8	·072	2	0

Adams and Joy, as well as by Edwards, in their determinations of absolute magnitudes, *i.e.* nebulous lines indicate low absolute brightness, while narrow and sharp lines are characteristic of the more luminous stars. The fact that this criterion is probably dependent upon rotational velocity is, of course, an interesting result which should be investigated further. A tentative explanation of this has been given elsewhere.*

* *The Observatory*, **54**, 80, 1931.

Our analysis of the spectra of giants and dwarfs shows that all single stars belonging to the later spectral classes show little rotation. On the other hand, a number of spectroscopic binaries of late type, such as W Ursae Majoris, have very rapid rotation. In fact, Adams and Joy have in a number of cases successfully predicted that stars of spectral class F or G showing diffuse lines are close spectroscopic binaries. It is probable that we have here a real difference in behaviour: in the early spectral types rapid axial rotation is observed in single stars about as frequently as in spectroscopic binaries; while in the later types rapid rotation occurs only in close binaries. This may have a bearing on the problem of the origin of double stars.

Yerkes Observatory:
1931 *March* 30.

On the Behaviour of Certain Simple Multiplets in Stellar Spectra.

By G. Shajn. (Plate 12.)

The study of the multiplet intensities in stellar spectra seems to be an urgent problem. The theoretical intensities given by the quantum theory are in agreement with the observations of laboratory emission spectra.* But in this case a divergence can arise, for instance, if there are conditions for a self-reversal.† As to the absorption spectra, so far as I am aware, there are no experimental tests.

The problem of contours and residual intensities of absorption lines in stellar spectra is so complicated in the mathematical, astronomical, and physical foundations that it is hardly possible to predict exactly the intensity ratio of lines in a multiplet. And yet this is probably easier to do for a close multiplet than for the lines of the Balmer series, which may be influenced by the interlocking effect,‡ Stark effect,§ dependence of transition from monochromatic to local thermodynamic equilibrium on the order number of lines in the same series,‖ etc.

The relative intensities of the lines in a multiplet depend mainly on the following factors: the transition probabilities, the formula for monochromatic absorption coefficient, the ratio of the line-absorption coefficient to the general one, the ratio of number of effective collisions to that of free transitions to the lower state of an excited atom.

The aim of this paper is to give some results with respect to certain simple multiplets in stellar spectra as they are brought out from the observations. To interpret the observed data it is necessary to consider some general theoretical points of the problem in question.

* L. de Kronig, *Zs. für Phys.*, **31**, 885, 1925; **33**, 261, 1925. Russell, *Proc. Nat. Acad. of Science*, **2**, 314, 1925. Ornstein and Burger, *Zs. für Phys.*, **31**, 355, 1925. Sommerfield and Höhnl, *Zitz. d. Preuss. Akad. d. Wiss.*, 141, 1925.
† Dorgelo, *Zs. für Phys.*, **31**, 827, 1925.
‡ Eddington, *M.N.*, **89**, 620, 1929.
§ O. Struve, *Ap. J.*, **69**, 173, 1929.
‖ Ambarzumian, *Zs. für Phys.*, **60**, 255, 1930.

§ 1. The few theoretical papers dealing with this problem give discordant results. Milne * believes that the theoretical ratio 2 : 1 should apply to the residual intensities of the absorption lines of Ca^+ formed by a thin layer of gas (" in absorption K should be twice as strong as H "). By integration through the different strata Milne finds that the same ratio should hold for a stratified mass. He derives that the ratio of residual intensities is equal to the ratio of statistical weights; that is, for an atmosphere the residual intensities obey the same rule as would be inferred from a straightforward application of the summation rules of Burger and Dorgelo.

It seems to me that according to new conceptions this result must be corrected. In fact, Milne gives

$$\left(\frac{1}{r'} - 1\right)\frac{2 + J'}{\alpha'} = \left(\frac{1}{r''} - 1\right)\frac{2 + J''}{\alpha''} \quad . \quad . \quad (1)$$

where r' and r'' are the residual intensities, $J = (e^{h\nu/kT} - 1)^{-1}$ and

$$\alpha\Delta\nu = \frac{c^2}{8\pi\tau\nu^2}\frac{q_2}{q_1} \quad . \quad . \quad . \quad . \quad (2)$$

The line of effective width $\Delta\nu$ is supposed to be sharply bounded (within this range the absorption coefficient is constant and equal to α). The notations are the usual ones.

Neglecting in (1) J in comparison with unity and using (2), we have a good approximation for deep lines of a doublet,

$$\frac{r_1}{r_2} = \frac{q_2''}{q_2'} \cdot \frac{\tau'}{\tau''} \cdot \frac{\Delta\nu'}{\Delta\nu} \quad . \quad . \quad . \quad . \quad (3)$$

Milne's result $\frac{r_1}{r_2} = \frac{q_2''}{q_2'}$ is based on the assumptions $\tau_1' = \tau''$ and $\Delta\nu' = \Delta\nu''$. The first assumption is quite correct since the average life $\tau' = \frac{1}{A'_{2\to1}}$ and $\tau'' = \frac{1}{A''_{2\to1}}$, and according to Ornstein and Burger † $A''_{2\to1} = A'_{2\to1}$.

But the assumption $\Delta\nu' = \Delta\nu''$ is in disagreement with the new theoretical and observational representations. In fact, the accurate photometric measurements of line contours ‡ bring out an important feature that in such doublets as those of Ca^+ and Na, for instance, $^2S - {}^2P_2$ is much wider than $^2S - {}^2P_1$. On the other hand, Unsöld's theory gives the widths of the lines in doublets to be proportional to $\sqrt{f}$, where f is the " oscillatory strength." A much better approximation will be reached if in (3) $\frac{\Delta\nu'}{\Delta\nu''}$ is replaced by $\sqrt{f_1/f_2}$.

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* *M.N.*, **86**, 8, 1926.

† *Loc. cit.*

‡ Unsöld, *Zs. für Phys.*, **44**, 793, 1927; **46**, 765, 1928; Payne, *Proc. Nat. Ac. of Sci.*, **14**, 399, 1928.

According to Ladeburg * it may be assumed that

$$f \propto B_{1\to 2} h\nu \frac{m}{\pi e^2} \quad . \quad . \quad . \quad . \quad . \quad (4)$$

Finally, for a doublet,

$$\frac{\Delta\nu'}{\Delta\nu''} = \sqrt{\frac{f_1}{f_2}} = \sqrt{\frac{q_2'}{q_2''}} \quad . \quad . \quad . \quad (5)$$

Inserting (5) in (3) we find that for the strong lines the residual intensities are inversely proportional to the square-roots of statistical weights,

$$\frac{r_1}{r_2} = \sqrt{\frac{q_2''}{q_2'}} \quad . \quad . \quad . \quad . \quad (6)$$

The corrected formula (6) gives a better approximation to the observations, but there remains still a considerable divergence, as will be shown later. But it is doubtful in general, whether the relation $\frac{r_1}{r_2} = \frac{q_2''}{q_2'}$ does hold in absorption spectra even for a thin layer of gas.

Interesting results may be reached if the problem in question be considered in the light of Eddington's formula for absorption lines. Let us take the "standard case" †

$$r = \frac{1 + \frac{2}{3}q}{1 + \eta + \frac{2}{3}q} \quad . \quad . \quad . \quad . \quad (7)$$

where

$$\eta = \frac{s_\nu}{\kappa_0} \quad \text{and} \quad q^2 = 3(1+\eta)(1+\epsilon\eta).$$

It is to be remembered only that ϵ is the part of the monochromatic absorption (line absorption) which for re-emission in the same frequency is lost by superelastic collisions.

Assuming κ_0 to be constant for all the lines in a close multiplet (*i.e.* P_ϵ constant) and using Unsöld's formula for s_ν ‡ we have

$$\frac{\eta_1}{\eta_2} = \frac{f_1}{f_2} \quad . \quad . \quad . \quad . \quad . \quad (8)$$

Considering instead of r the absorption fraction $1 - r$ we find for the absorption ratio

$$\frac{1-r_1}{1-r_2} = \frac{f_1}{f_2} \cdot \frac{1 + \eta + \frac{2}{3}\sqrt{3(1+\eta)(1+\epsilon\eta)}}{1 + \frac{f_1}{f_2}\eta + \frac{2}{3}\sqrt{3\left(1+\frac{f_1}{f_2}\eta\right)\left(1+\frac{f_1}{f_2}\epsilon\eta\right)}} \quad . \quad (9)$$

For an illustration let us take two doublets where $\frac{\eta_1}{\eta_2} = \frac{f_1}{f_2} = 2$; let

* *Zs. für Phys.*, **4**, 454, 1921.

† *M.N.*, **89**, 623, 1929. It is obvious that the application of Eddington's model to the lines within a multiplet has to some extent a conditional character, but it appears that the rough assumptions with respect to the ratio $\frac{S\nu_0'}{S\nu_0''}$ and to the constancy of κ_0 as a first approximation are not too violent.

‡ *Loc. cit.*

η be 3 and 15 for the second components respectively. Then, from (9) we find in the case $\epsilon = 0$ the central absorption ratios 1·26 and 1·10 respectively. If $\epsilon = 0{\cdot}2$, the other things being the same, we have for the absorption ratios, 1·20 and 1·05. Using the same numerical data we find for the intensity ratio the values 1·18 and 1·08 respectively. Therefore the contrast between the central absorption of lines within a multiplet must be rather small. In addition, the contrast is probably smaller for the stronger lines. Of course, the formula (9) corresponds to a very rough scheme, which may be modified to some extent if account be taken of the difference in mean depth of the origin of lines within a multiplet and of the approximate character of the assumption (8) that the absorption line coefficients are proportional to their "oscillatory strengths." However, the relation (9) accounts, as will be seen later, for the essential features of multiplets in stellar spectra.

Unsöld,* taking account of the collisions for the radiating atoms, derives a formula for the central intensity depending only on $\lambda = \frac{2\kappa}{s}$, which is essentially Eddington's ϵ. On the basis of an unpublished investigation by Fues showing that $\frac{\kappa}{s}$ is constant for all the lines within a close multiplet, Unsöld concludes that their intensities will be equal. This is the other extreme point of view in comparison with $\frac{r_1}{r_2} = \frac{q_2''}{q_2'}$ or even $\frac{r_1}{r_2} = \sqrt{\frac{q_2''}{q_2'}}$.

It is of interest to note that Unsöld's conclusion may be derived from Eddington's equation (7), if η is very large. Then $q = \eta\sqrt{3\epsilon}$ and

$$r = \frac{\sqrt{3\epsilon}}{\frac{2}{3} + \sqrt{3\epsilon}} \quad . \quad . \quad . \quad . \quad (10)$$

It is evident that we may assume ϵ to be constant for all the lines within a multiplet. But this conclusion, as well as Unsöld's, is true if the general absorption may be neglected, and, on the other hand, if all the lines of a multiplet originate at the same depth. We cannot be sure that this is the case.

It is advantageous to consider the total line absorption instead of the central one. Let us take the Schwarzschild-Schuster formula for the radiation:

$$r = \frac{0{\cdot}5 + \cos i}{1 + \sigma\mathrm{H}} + \frac{e^{-\sigma\mathrm{H}\sec i}(0{\cdot}5 - \cos i)}{1 + \sigma\mathrm{H}} \quad . \quad . \quad (11)$$

A fair approximation for a stellar disc $i = 0{\cdot}5$ leads to the known formula

$$r = \frac{1}{1 + \sigma\mathrm{H}} \quad . \quad . \quad . \quad . \quad (12)$$

where $\sigma\mathrm{H} = \int \sigma\rho ds$. The notation is as usual.

* *Festschr. zum 60 Geburtstage A. Sommerfelds*, 95.

For the full line absorption ϵ we have

$$E = \int_{-\infty}^{+\infty} \frac{\sigma H}{1 + \sigma H} \quad . \quad . \quad . \quad . \quad (13)$$

In the case of damping radiation we can use Unsöld's formula for σ,

$$E = \frac{2\pi e^4 \lambda_0^2 N f}{3m^2c^4(\lambda - \lambda_0)^2} \quad . \quad . \quad . \quad . \quad (14)$$

Writing

$$K = \frac{2\pi e^4 \lambda_0 N H}{3m^2c^4} \quad . \quad . \quad . \quad . \quad . \quad (15)$$

we find

$$E = \int_{-\infty}^{+\infty} \frac{K f d\lambda}{Kf + (\lambda - \lambda_0)^2} \quad . \quad . \quad . \quad (16)$$

The integration leads to

$$E = \sqrt{Kf} \arctan \frac{\lambda - \lambda_0}{\sqrt{Kf}} \quad . \quad . \quad . \quad (17)^*$$

Since according to Unsöld's theory $\lambda - \lambda_0$ is proportional to $\sqrt{f}$, we have for the full absorption ratio in a doublet,

$$\frac{E_1}{E_2} = \sqrt{\frac{f_1}{f_2}} = \sqrt{\frac{q_2'}{q_2''}} \quad . \quad . \quad . \quad . \quad (18)$$

When this investigation was ready, Struve and Elvey † published a paper in which is given the same relation derived in a different manner.

The question of the theoretical prediction of the full absorption ratio within a multiplet depends on the form of the absorption or scattering coefficient. Any other formula for σ in (14) will give a different result. Unsöld ‡ derives a formula for shallow lines which gives the full line absorption to be proportional to f.

If the collision effect be included the problem becomes more complicated. This effect generally decreases with increasing distance from the line centre. The physical reason for this, according to the general idea, seems to be that when an atom absorbs light wide of the line centre the re-emission follows much more quickly than for a central excitation.§ If so, the de-excitation by collision is reduced. On the other hand, the general absorption increases for the wing. It may be supposed that the contour becomes more shallow. The problem, as connected with difference in the optical depths between the line centre and the wings, requires a special investigation. Probably when collision effect is taken into account the full absorption ratio becomes smaller than that given by (18).

Summing up these general considerations, based on some rough

* In deriving this formula I was following a suggestion by Ambarzumian. Later I found that Russell, in *Ap. J.*, **70**, 11, 1929, also gives the total absorption to be proportional to $\sqrt{f}$.

† *Ap. J.*, **72**, 267, 1930.

‡ See Eddington, *M.N.*, **89**, 635, 1929.

§ *Zs. f. Phys.*, **59**, 364, 1930.

approximations, we can say that at present, owing to the physical, astronomical, and mathematical complications, the theory cannot predict exactly the intensity or absorption ratio for the centres of lines belonging to the same multiplet.* But some interesting features may be emphasized to be tested by the observations.

The most important feature consists in much smaller contrast between the absorption lines within a multiplet in stellar spectra in comparison with that observed in the laboratory emission ones. For this reason it is not surprising to observe in stellar spectra the faintest line of the multiplets. Further, a difference may be suggested between the strongest and the faint multiplets if the rôle of effective collisions for radiating atoms in comparison with the free transitions becomes important, the contrast being smaller within a strong multiplet.

The total absorptions of lines within a multiplet in the case of damping radiation are proportional to the square roots of the statistical weights. If the collision effect be taken into account the contrast between the full-line absorption in a multiplet is probably smaller than that given by the square-root relation.

It may also be expected that the collision effect as connected with the effective target of atoms will change with increasing excitation potential. In fact, Gerasimovič,† using a method worked out by R. H. Fowler, gives some numerical results which show that the rôle of collision ionization for higher quantum states is much larger than that for the lower ones. For this reason the study of the multiplet intensities as functions of the excitation potential becomes urgent as to the intensity itself as well as to the intensity ratio within a multiplet.

Finally, it is of interest to investigate the multiplet intensities along the spectral sequence for giants as well as for dwarfs, in order to see whether the intensity ratio changes with increasing temperature and luminosity.

§ 2. *Observations.*—The results of observations of doublets of *Na*, Ca^+ and of triplets of *Mg* (sharp and diffuse series) are given in this preliminary account. For the sake of brevity the detailed description of the photometric procedure used will be given elsewhere.

The spectrograms were taken with a single-prism spectrograph attached to the 40-inch reflector made by Sir Howard Grubb (linear dispersion $1^{mm} = 36$ A. at $H\gamma$). Two methods for standardizing the plates were used. One consisted in obtaining the spectrum of an Osram lamp the light of which passed through a small wedge placed in front of the slit of the spectrograph. On the wedge were traced very narrow lines to give marks on the characteristic curve. Previously we have used a step weakener of Zeiss, but owing to the small opacity range and the non-uniformity we have now prepared a wedge ourselves. The other method consisted in the variation of the slit width. Often both methods were used for the same set of stellar spectrograms. The agreement between the two methods is found to be satisfactory. The

* A paper by A. Pannekoek in *M.N.*, **91**, 139, 1930, promises to give a closer approximation, but it appeared when this study was ready.

† *H.B.*, **857**, 17, 1928.

usual precautions in photographic photometry were taken to obtain the results as accurately as possible. The plates used for the visual region were Ilford Panchromatic and Wratten & Wainwright Panchromatic, and for the photographic region Ilford Monarch.

The traces were taken with a self-registering Koch's microphotometer. The results of observations are given in Tables I. to IV., which are self-explanatory.

TABLE I.

Na (1^2S-1^2P).

Star.	Date.	Central Absorption. D_1.	Central Absorption. D_2.	Intensity Ratio. $D_2 : D_1$.
α Cyg.	1930 Apr. 16	18	15	
	28	20	18	
		19_0	16_5	1·03 : 1·00
α CMi.	1929 Nov. 21	16	14	
	1930 Feb. 12	16	11	
	12	13	10	
	Mar. 9	19	15	
	9	17	16	
	9	18	17	
	Apr. 24	12	12	
	24	15	11	
		$15_7 \pm 0_8$	$13_2 \pm 0_9$	1·03 : 1·00
β Cas.	1930 Aug. 2	15	12_5	1·03 : 1·00
α Per.	1930 Feb. 24	18	15	
	25	17	16	
	Apr. 16	26	23	
		$20_3 \pm 2_8$	$18_0 \pm 2_5$	1·03 : 1·00
γ Cyg.	1929 Oct. 30	20	18	
	30	20	19_5	
	30	20	16	
	1930 Nov. 9	20	15	
	19	27	28	
		$21_4 \pm 1_7$	$19_3 \pm 2_5$	1·03 : 1·00
Moon	1929 Oct. 22	25	22	
	22	25	23	
	1930 Oct. 8	28	21	
	8	30	25	
	8	28	23	
		$27_2 \pm 1_0$	$22_8 \pm 0_7$	1·06 : 1·00

TABLE I.—*continued.*

Na ($1^2S - 1^2P$).

Star.	Date.	Central Absorption. D_1.	Central Absorption. D_2.	Intensity Ratio. $D_2 : D_1$.
α Aur.	1929 Nov. 21	24	18	
	21	31	30	
	21	26	24	
	1930 Feb. 12	24	20	
	12	27	24	
	12	27	23	
		$26_5 \pm 1{\cdot}0$	$23_2 \pm 1_7$	1·04 : 1·00
β Gem.	1929 Oct. 28	29	25	
	28	29	26	
	1930 Feb. 24	28	25	
	25	30	24	
		$29_0 \pm 0_4$	$25_0 \pm 0_4$	1·06 : 1·00
α Boo.	1929 Apr. 30	37	31	
	30	32	26	
	May 7	38	31	
	1930 Feb. 23	30	24	
	July 7	30	27	
		$33_4 \pm 1_7$	$27_8 \pm 1_2$	1·08 : 1·00
α Ari.	1930 Nov. 9	38	33	
	Dec. 9	49	44	
	9	47	39	
		$44_7 \pm 3_4$	$38_7 \pm 3_2$	1·11 : 1·00
α Tau.	1929 Oct. 23	42	36	
	Nov. 17	53	46	
	17	53	47	
	1930 Feb. 23	49	40	
		$49_2 \pm 2_6$	$42_2 \pm 2_6$	1·14 : 1·00
δ Oph.	1930 Aug. 4	48	44	
	13	44	40	
		46_0	42_0	1·07 : 1·00

TABLE I.—*continued.*

Na ($1^2S - 1^2P$).

Star.	Date.		Central Absorption. D_1.	Central Absorption. D_2.	Intensity Ratio. $D_2 : D_1$.
β And.	1929 Oct.	30	46	39	
		30	43	38	
	Nov.	17	40	36	
		17	49	46	
	1930 Nov.	19	50	44	
			$45_6 \pm 1_8$	$40_6 \pm 1_9$	1·09 : 1·00
α Ori.	1929 Oct.	16	47	47	
		18	42	41	
	1930 Feb.	12	46	46	
		12	45	44	
			$45_0 \pm 1_1$	$44_5 \pm 1_3$	1·01 : 1·00
α Sco.	1929 Apr.	26	50	46	
	1930 Apr.	26	48	43	
		26	47	45	
	July	23	53	51	
		23	50	47	
			$49_6 \pm 1_0$	$46_4 \pm 2_6$	1·06 : 1·00
α Her.	1929 July	1	48 ?	48 ?	
		7	46 ?	44 ?	
			47 ?	46 ?	

TABLE II.

Mg ($1^3P - 1^3S$).

Star.	Date.		Central Absorption. 5184.	5173.	5167.	Intensity Ratio. 5173 : 5184.
α CMi.	1930 Feb.	12	25	26	24	
	Mar.	9	27	26	25	
		9	26	23	27	
		9	28	23	30	
	Apr.	24	27	26	26	
		24	24	24	28	
			$26_2 \pm 0_6$	$24_7 \pm 0_6$	$26_7 \pm 0_7$	1·02 : 1·00

Table II.—*continued.*

Mg ($1^3P - 1^3S$).

Star.	Date.	Central Absorption. 5184.	5173.	5167.	Intensity Ratio. 5173 : 5184.
β Cas.	1930 Aug. 2	19	18	18	1·01 : 1·00
α Per.	1930 Feb. 25	21	22	31	
	Apr. 16	19	18	30	
	16	17	18	30	
		$19 \pm 1_1$	$19_3 \pm 1_3$	$30_3 \pm 0_3$	
γ Cyg.	1929 Nov. 19	30	33	39	
	19	27	34	40	
	1930 Nov. 9	28	27	39	
	19	29	31	42	
		$28_5 \pm 0_7$	$31_2 \pm 1_5$	$40_0 \pm 0_7$	
Moon	1929 Nov. 19	43	39	37	
	Dec. 5	52	48	42	
	5	50	46	39	
	1930 Oct. 8	48	44	43	
	8	46	43	39	
		$47_6 \pm 1_6$	$44_0 \pm 1_5$	$40_0 \pm 1_1$	1·07 : 1·00
α Aur.	1929 Nov. 21	43	38	42	
	1930 Feb. 12	43	40	45	
	12	49	36	41	
		$45_0 \pm 2_0$	$38_0 \pm 1_2$	$42_7 \pm 1_2$	1·13 : 1·00
β Gem.	1929 Oct. 28	46	54	51	
	Dec. 25	46	43	50	
	25	48	49	46	
		$46_7 \pm 0_7$	$48_7 \pm 3_2$	$49_0 \pm 1_5$	
α Boo.	1929 Apr. 30	52	52	48	
	1930 Feb. 23	54	53	50	
	July 7	57	57	50	
	9	52	50	46	
	11	59	59	55	
		$54_8 \pm 1_4$	$54_2 \pm 1_7$	$49_8 \pm 1_4$	

TABLE II.—*continued.*

Mg ($1^3P - 1^3S$).

Star.	Date.	Central Absorption. 5184.	5173.	5167.	Intensity Ratio. 5173 : 5184.
α Ari.	1930 Nov. 9	51	52	50	
	9	52	49	45	
	Dec. 5	56	51	48	
	5	56	50	47	
	9	62	56	55	
	9	60	57	53	
		$56_2 \pm 1_7$	$52_5 \pm 1_3$	$49_7 \pm 1_5$	
ε Peg.	1930 Aug. 13	52	58	58	
	Dec. 5	47	53	53	
	5	49	52	50	
		$49_3 \pm 1_4$	$54_3 \pm 1_9$	$53_7 \pm 2_3$	
α Tau.	1929 Nov. 17	55	53_5	52	
	17	55	54	52	
	1930 Oct. 2	54	59	56_5	
		$54_7 \pm 0_3$	$55_5 \pm 1_8$	$53_5 \pm 1_5$	
δ Oph.	1930 Aug. 13	51	53	51	
β And.	1929 Nov. 17	62	63	61_5	
	1930 Nov. 19	58	62	60	
		60_0	62_5	60_8	
α Ori.	1929 Oct. 16	57	73	70_5	
	18	54	66	71	
	Nov. 13	55	66	69_5	
	13	58	70_5	70_5	
	1930 Nov. 8	59	64_5	67	
		$56_6 \pm 0_9$	$68_0 \pm 1_6$	$69_6 \pm 0_7$	
α Sco.	1929 July 12	53	63	66	
		54	58	66	
		53_5	60_5	66	
β Peg.	1930 Aug. 13	59	66	64	

TABLE III.

Mg (1^3P-2^3D).

Star.	Date.	Central Absorption. 3838.	3832.	3829.	Intensity Ratio. 3829 : 3832 : 3838.
α Cyg.	1929 July 6	31	27	14	
	6	28	27	16	
		29_5	27_0	15_0	1·21 : 1·03 : 1·00
α CMi.	1929 Nov. 11	69	67	64	
	Dec. 5	63	64	57	
	1931 Mar. 1	67	66	58	
	1	66	65	57	
		$66_2\pm1_2$	$65_5\pm0_7$	$59\pm1_7$	1·21 : 1·02 : 1·00
Moon	1930 Oct. 8	78	78	74	
	8	75	74	72	
	Nov. 5	77	76	66	
	5	73	72	67	
		$75_7\pm1_1$	$75_0\pm1_3$	$69_7\pm1_9$	1·25 : 1·03 : 1·00
β Gem.	1929 Oct. 31	72	71	65	
	Nov. 23	70	65	58	
	1930 Nov. 11	78	76	68	
		$73_3\pm2_4$	$70_7\pm3_2$	$63_7\pm3_9$	1·36 : 1·10 : 1·00
α Boo.	1930 July 3	71	72	66	
	Aug. 13	73	69	61	
	15	72	72	68	
		$72_0\pm0_6$	$71_0\pm1_0$	$65_0\pm2_1$	1·25 : 1·04 : 1·00
α Tau.	1930 Oct. 30	66	64_5	55	1·32 : 1·04 : 1·00
α Ori.	1930 Nov. 8	63	63	59	
	9	62	66	56	
		62_5	64_5	57_5	1·13 : 0·95 : 1·00

TABLE IV.

Ca^+ (1^2S-1^2P).

Star.	Date.	Central Absorption.* K.	Central Absorption.* H.	Intensity Ratio. H : K.	Full Absorption.† K.	Full Absorption.† H.	Ratio. K : H.
Moon	1930 Nov. 5	92	92		14·03	11·45	$1·22_5$
	5	90	89		12·54	9·50	1·32
	9	89	87		14·48	11·63	$1·27_5$
	9	89	91		13·40	10·46	1·28
		$90_0 \pm 0_7$	$89_8 \pm 1_1$	1·02	13·62±·42	10·76±·54	1·27±·02
β Gem.	1929 Nov. 23	89	90		14·75	11·59	1·27
	1930 Oct. 31	92	92		17·65	14·53	$1·21_5$
	Nov. 11	92	91		15·61	12·27	$1·25_5$
		$90_3 \pm 1_1$	$91_0 \pm 0_6$	0·93	16·00±·86	12·80±·89	1·25±·02
α Boo.	1930 July 3				16·29	12·54	1·30
	5				17·56	13·85	1·27
	7				17·52	13·40	1·31
	Aug. 13				15·01	12·09	1·24
	15				16·75	13·26	1·26
	1931 Feb. 1				18·06	12·94	1·39
					16·86±·45	13·01±·26	1·30±·02
α Ori.	1930 Nov. 8	92	92				
	8	92_5	93				
	9	91	92				
		$91_8 \pm 0_4$	$92_3 \pm 0_3$	0·94			

The results of this paper are based on the study of the central line absorption of members of the principal series of *Na* and Ca^+, diffuse and sharp (partly) series of *Mg*. As to the total absorption, only Ca^+ may be considered with certainty. It is true that in Ca^+ reversal is to be expected, but it is known that certain stars are free from this effect. At the longest exposure, when H and K were highly over-exposed, I could not find any direct trace of reversal in β Geminorum. In any case the reversal affects mostly the central part of the line contour and but little the full absorption.

Though we have measured the absorption fully for *Na* and partly for *Mg*, I thought it impossible to compare them with the theoretical multiplet predictions without a study of the instrumental (both spectrographic and microphotometric) and photographic errors, and the

* The intensity 100 is adopted for the background at 3993·5 A.

† In angstroms.

blending effect. But these uncorrected full absorptions represent an important characteristic for other purposes (in § 6 of this paper they are used in application to the ionization theory).

§ 3. There are many factors influencing the contours of lines, but dealing here with the central absorption it is necessary to consider, at least, the effect of dispersion and the blending effect. There is no doubt that the intensities and contours of lines depend on the dispersion used, unless the lines are very wide. For this reason it is to be feared that the observed ratio of multiplet lines is so affected that the comparison with any theory is unsound. Having no intention to consider here the problem in detail, we must, however, take account of this effect. It is obvious that only the differential effect of dispersion is of value for the multiplet intensity problem. The first step is to compare the observed intensity ratio of lines in spectra obtained with different dispersion. Such an investigation will be published elsewhere. Here is given only a preliminary formula which is based on the comparison of 40 lines (about λ 4340) in the solar spectrum obtained with three- and one-prism spectrographs (linear dispersions are 1 mm. = 8·8 A. and 36 A. respectively):

$$A_{III} = 7{\cdot}20 + 1{\cdot}41_3 A_I - 0{\cdot}006_5 A_I^2 \quad . \quad . \quad (19)$$

A_{III} and A_I are the central absorptions respectively. This approximate formula, being in dependence on the slit widths used in both spectrographs, is of particular interest. In addition, it is to be noted that a better approximation may be reached if terms depending on the line width and the wave-length are included. The formula (19) allows us to reduce the observed data as they are given in Tables I. to IV. to the three-prism dispersion, and therefore the dispersion effect will be reduced. But it is evident at once that in our case the reduction is very small. For instance, the observed intensity ratio D_2/D_1 (Table I.) for α Tauri is 1·14, and the reduced one by (19) is 1·13. Certainly, in the case of large intensity difference of lines this effect becomes sensible. For this reason we may be sure that the intensity ratios as they are given in Tables I. to IV. may be affected very little, at least, in comparison with the three-prism dispersion.

Other evidence may be found in the comparison with some recent measures of the lines in solar spectra taken with very large dispersion. Unsöld does not give for *Na* the intensities themselves, but from fig. 3 of his paper * it may be seen that the intensity ratio is of the same order as that given here in Table I. The same is also evident from plate 3 in Carroll and Moss's paper.†

With the moderate dispersion used in stellar spectroscopy the majority of lines are to some extent blends, and it is necessary to consider whether the lines in question are free from this effect. Since with the dispersion used D_1 and D_2 are well separated, the influence may be expected from the lines separated by $\pm$ 15 A. As to *Na* lines, we may be sure that they are free from blending. As far as this can be judged on the basis of the Rowland Revision there are only a few faint

* *Loc. cit.*

† *M.N.*, **91**, 191, 1930.

lines in the solar spectrum which may affect the intensity of D_1 and D_2. This probably holds also for other spectral types since the blending lines are mostly of telluric origin.

The diffuse triplet of *Mg* and the doublet of Ca^+ may be also considered as free from blending. Unfortunately this is not the case for the sharp triplet of *Mg*, and the last must be used here with great precaution, and only within a small range of the spectral classes. Since this triplet is very conspicuous in stellar spectra it is worth while to give a detailed account of the blending effect, which will be also useful for other problems.

From the Rowland Revision* we find the blending lines as follows:—

Table V.

	Int.	Series.	λ.	Int.	Series.	λ.	Int.	Series.
·96	−3		5171·0			5165·42	2	
·35	−3		5171·02	−3		5·58	−3	
·60	−3		1·61	6	*Fe* ($1^3F-1^3F'$)	5·97	−3	
·74	−2		2·22	−1		6·29	3	*Fe* ($1^5D-1^7D'$)
3·62	30	*Mg* (1^3P-1^3S)	2·70	20	*Mg* (1^3P-1^3S)	7·33	15	*Mg* (1^3P-1^3S)
4·20	−2		3·33	−3		7·51	5	*Fe* ($1^3F-1^3D'$)
4·27	2	*Fe*	3·49	−2		7·72	−1	
4·56	1		3·75	2	*Ti* ($1^3F-1^3F'$)	7·96	−3	
4·84	−3		3·91	−3		8·19	−1	
5·04	−3		4·04	−2		8·66	1	*Fe* ($1^5D-1^7D'$)
			4·43	−3		8·91	3	

Taking into account the dispersion used, we see that the central absorption of 5184 may be considered as nearly free from blending; as to other members, 5173 is partly and 5167 is highly blended.

It is of interest to know whether this blending effect changes in other spectral classes. This can be determined if we study the intensity of other lines belonging to the same multiplets as the blends in question, mainly 5166·29, 5167·51, and 5171·61. Such a short investigation was performed, and the preliminary results (central absorption), based on small material, are given in Table VI.

Table VI. shows that the blending effect for the lines in question increases in late-type stars, and also in *c*-stars such as γ Cygni. It explains at once the peculiarities in the relative intensities of lines in the sharp *Mg* triplet in stars of different spectral types and luminosities, as this is brought out in Table II.

A conclusion may be derived that only two out of three lines of *Mg* (5184 and 5173) may be used, and those with some caution, in G-type stars (except super-giants). As to the study of the behaviour of the sharp triplet of *Mg* along the spectral sequence, there remains the sole line

* C. E. St. John, C. E. Moore, L. M. Ware, E. T. Adams, N. D. Babcock, *A Revision of Rowland's Preliminary Table.*

TABLE VI.

	Fe ($1^3F-1^3D'$).		*Fe* ($1^5D-1^5D'$).		*Fe* ($1^3F-1^3F'$).		*Ti* ($1^3F-1^3F'$).			
C.Mi.	5227·19.	5270·36.	5110·41.	5204·53.	5194·95.	5216·28.	5147·48.	5152·18.	5192·98.	5210·39
α CMi.	21	18	..	11	12	11	..	..	..	9
β Cas.	11	18	..	14	..	..	..	..	..	14
α Per.	23	17	...	..	..	..	..	..	..	..
γ Cyg.	35	37	..	..	22	18	..	22	22	..
Moon	30	32	..	19	13	15	..	10	19	19
α Aur.	33	34	10	20	19	15	..	14	21	21
β Gem.	30	39	15	22	19	15	..	17	27	21
α Boo.	28	40	17	25	17	10	..	18	22	30
α Ari.	42	51	22	34	28	21	..	22	34	34
ε Peg.	46	51	27	40	27	22	30	30	42	39
α Tau.	38	49	24	49	25	20	29	29	38	50
δ Oph.	39	50	..	48	..	15	..	..	39	51
β And.	47	60 ?	40 ?	36 ?	38	16	..	..	..	41
α Ori.	47	54	39	52	36	22	37	37	46	48
α Sco.	45	50	35	48	35	21	32	32	45	44

5184. It must be emphasised that the weight of the triplet in question in this investigation is very small.

Before further discussion it is to be mentioned that within the region of H and K lines the plate sensitiveness and the instrumental and atmospheric absorption change very appreciably. The observed continuous background must be corrected for the differential effects in question. The procedure followed consisted in adopting a black-body intensity distribution within the region under consideration. Of course, an uncertainty remains with regard to the assumed temperature, but within reasonable limits the results change but little. Another uncertainty arises from a partial overlapping of the wings of H and K, and this effect cannot be accounted for. But bearing in mind that only the ratio of full absorptions is of interest for the problem in question, we must conclude that this error cannot exceed a few per cent. of the ratio itself.

§ 4. *Results.*—The most noticeable feature of the multiplets as it is brought out from the observations (Tables I. to IV.) is the very small difference in central intensities of the lines belonging to the same multiplet. The contrast between absorption lines within a multiplet in stellar spectra is much smaller than that observed in the laboratory emission spectra. This peculiarity as it is shown in § 1 may be deduced from the equation for an absorption line, taking into account the collision effect. An analogy must be noted between this feature of multiplets and the very slow decrease of central intensity of the Balmer lines and other series with increasing order numbers.

It seems that the slow decrease of the line intensity in a multiplet and the Balmer series has an intimate relation to the known excess of the numbers of atoms for the higher members of the Balmer series

computed on the basis of the observed contours of lines. If the equation similar to (9) may have an application to the contour of a line in general, it appears to be possible to find a sequence of the line contours within a multiplet which, being interpreted in the light of Unsöld's theory, will give an excess of atoms for the higher members. This suggestion requires a special study. It is of interest to note that a paper by R. van der R. Woolley * brings out an excess of atoms for the fainter lines of a multiplet. If so, the observed small contrast in central intensity and the excess of atoms for the fainter lines represent a phenomenon quite similar to that observed in the Balmer series.

As to the difference between the strong and faint multiplet, or between the stronger and fainter lines within a multiplet, the observational data are too few for a definite conclusion. It may be emphasised (Table III.) that in the diffuse triplet of *Mg* the contrast between the two brighter lines is much smaller than that between the second and third line. The mean intensity ratio for 7 stars is $1{\cdot}24_7 : 1{\cdot}03_0 : 1{\cdot}00$. The multiplet formula gives 1 : 3 : 5 and the laboratory emission spectrum 1·00 : 2·84 : 4·65.†

The central intensities of the strong lines of H and K in late-type stars show a peculiarity to be noted: the intensity ratio $\frac{H}{K}$ is often smaller than unity or near to it even in stars showing no evident reversal (Table IV.). This may be interpreted either as an indication of the presence of indirect reversal, or as a real feature of very strong multiplets.

The other feature to be noted consists in a small but real difference between the same multiplets in spectra of individual stars. As an illustration the comparison of *Na* in α Orionis and α Tauri shows that the intensity difference in the former is quite negligible. This relates to the central intensity but not to the full one. In Plate 12 (figs. 2 and 3) the observed traces are given. This is not an occasional defect, for the spectrograms obtained on different days bring out the same feature.‡ This cannot be attributed to the difference in the spectral classes (compare α Orionis and α Scorpii). Perhaps there is an analogy between the origin of *Na* and Ca^+ lines in such stars as α Orionis.

Notwithstanding these small differences within a spectral class there is brought out a constancy in the intensity ratio of lines within a multiplet along the whole spectral sequence. This peculiarity must be attributed to the uniformity of conditions in the origin of lines along the spectral sequence, probably to the very small range in the electron-pressure, controlling the general absorption and the part of line absorption due to the collision effect. It is to be emphasised that the considerable intensity increase of lines of *Na* and *Mg* from types F to M is not accompanied by any sensible change in the intensity ratio of lines within a multiplet. One must have in view this fact in the

* *Ap. J.*, **72**, 256, 1930.

† Dorgelo, *Physik. Zeits.*, **26** 756, 1925.

‡ We must keep in mind a possible influence of titanium oxide in this region.

consideration of the theoretical predictions with respect to the difference between the strong and faint multiplets.

As to the luminosity effect, we have no data for judgment since all the stars under consideration are giants, excluding the sun. Perhaps a very small difference may be suggested between the super-giants and giants, the contrast of the lines within a multiplet in the former being smaller. The striking example is α Orionis.

For the reasons mentioned in § 2 we consider here the full absorption only in the wide doublet of Ca^+, which is probably free from the instrumental and photographic errors. There is to be feared only the effect of reversal, which is very apparent in such stars as α Orionis and α Tauri. For this reason these stars are excluded. As to α Bootis, its reversal is limited to the central part of the contour and has but little influence on the full absorption ratio. Table IV. brings out a sensible deviation not only from the formula $\frac{E_1}{E_2} = \frac{q_2'}{q_2''}$ but even from $\frac{E_1}{E_2} = \sqrt{\frac{q_2'}{q_2''}}$, as given in this paper. The mean $\frac{E_1}{E_2}$ instead of the predicted $1{\cdot}41_4$ is $1{\cdot}27 \pm {\cdot}02$, $1{\cdot}25 \pm {\cdot}02$, and $1{\cdot}30 \pm {\cdot}02$ for the Moon, β Gem., and α Bootis respectively. The effect of reversal, if it holds in β Gem. at all, hardly suffices to account for this deviation. On the contrary, the effect of collision, as was mentioned above, leads probably to a decrease of the value for the full absorption ratio. The small contrast between the lines within a multiplet, as it is brought out from the study of the central and full absorption, explains the presence of the faintest lines of multiplets in stellar spectra. A similar conclusion is reached by Struve and Elvey * with regard to the full absorption of lines, though for *Si* III the contrast is found to be larger than for Ca^+. For such a conclusion there are theoretical foundations as illustrated in § 1.

Further progress may be reached by the accumulation of observations relating to multiplets of various strengths and different excitation potentials. Such a study is now in progress. This may be useful for the refinement of the theory.

§ 5. It is of interest to see whether the emission spectra in stars agree with the predictions given by the quantum theory. There are but few opportunities to test this in stellar spectra. In this paper we consider the emission of H and K in late-type stars, α Tauri and α Bootis, though we keep in mind that this is not quite a reliable example owing to the complex character of the reversal. Leaving for another paper the study of the reversal phenomenon, its asymmetry and other peculiarities, we are dealing here only with the central intensities. An illustration of the emission in α Tauri is reproduced in Plate 12 (fig. 1).

In Table VII. are given the ratios $\frac{K}{H}$ as they are derived from the corrected contours of H and K to make them comparable. The results depend to some extent on the adopted temperature (see § 3). For this reason two sets of the intensity ratio for α Tauri are given. The

* *Loc. cit.*

Fig. 1.

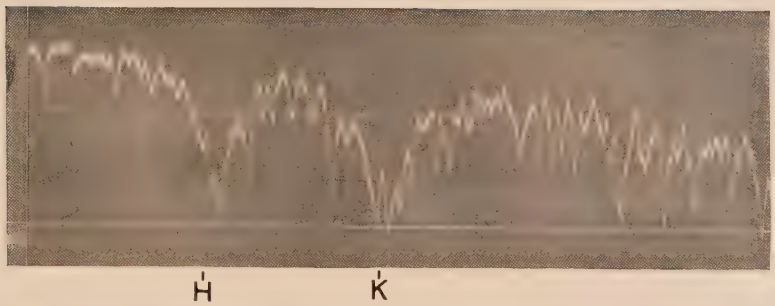


Fig. 2.

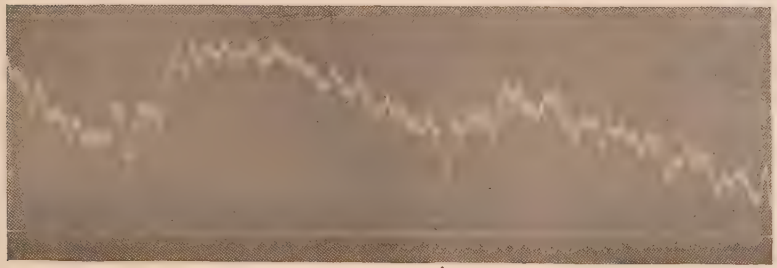


Fig. 3.

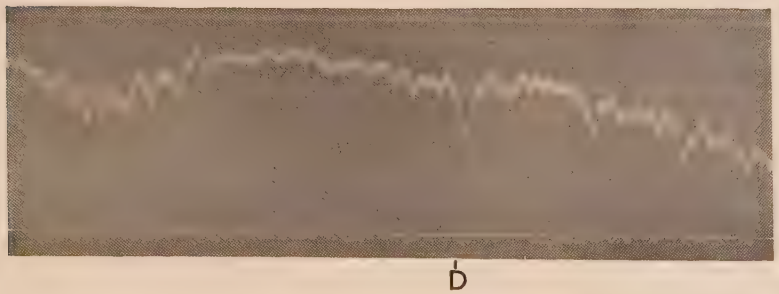


Fig. 1.—α Tauri, Region H and K.
Fig. 2.—α Orionis, Region near D.
Fig. 3.—α Tauri, Region near D.

weight of spectrograms for α Bootis is much smaller than for α Tauri, since they are slightly under-exposed to obtain the emission. The large discrepancies for α Bootis are probably due to this cause.

TABLE VII.

	$\frac{K}{H}(T=4000°)$ in emission.	$\frac{K}{H}(T=3000°)$ in emission.
α Tauri 1930 Sept. 21	1·51	1·47
Oct. 31	1·49	1·45
	1·50	1·46
α Bootis 1930 July 3	1·27	
7	1·34	
Aug. 15	1·18	
1931 Feb. 1	1·50	
	1·32	

We cannot say *a priori* that the observed ratio must be the same as that given by the multiplet formulæ. The measure of the agreement between them may give some idea of the optical depth for the lines under consideration. Table VII. brings out a sensible deviation from the theory, though in comparison with absorption lines the approximation to the laboratory emission spectra is much better. A similar correlation, as is known, is observed in Balmer emission lines in such stars as γ Cas.

In general these observations may be interpreted as an indication that the optical depth for emission in H and K is considerable since the self-reversal is probably connected with the increase of the optical depth. Perhaps it is of interest to note that for the bright lines H and K in the spectrum of the high level chromosphere (obtained during eclipse) there was found a satisfactory agreement between the observations and the predictions of the formulæ for multiplets.*

§ 6. I should like to conclude this paper by a brief discussion of the observational data for *Na* and *Mg* in the light of the theory of ionization. We cannot use Milne's generalization of the theory as it is given in a recent series of papers,† since the contours of narrow lines at moderate dispersion are to some extent spurious. Until the errors affecting the contours are specially studied, one has no recourse but to use the Fowler-Milne modification of the ionization theory.‡

In addition to central absorption we used also the full absorptions for *Na* and *Mg*, which are expressed in angstroms. For the reasons mentioned above, I am not sure that these measurements may be treated as relating to the real "full absorptions," though it is sur-

* Davidson, Minnaert, Ornstein, and Stratton, *M.N.*, **88**, 536, 1928.

† *M.N.*, **89**, 17, 157, 1928.

‡ *Ibid.*, **83**, 403, 1923; **84**, 499, 1924.

prising to me to find a fair general agreement with some measurements secured by Minnaert and v. Assenbergh with large dispersion (2A = 1^{mm}). For Rowland's intensities 10 and 15 Minnaert and v. Assenbergh * give 0·65 and 1·04 respectively, whereas for three lines in the solar spectrum, 5184 (35), D_1 (30), and D_2 (20), we find 1·65, 1·07, and 0·84 respectively. In Table VIII. are given the measured full intensities.

Table VIII.

	Full Absorption.				Full Absorption.		
Star.	D_1 5896.	D_2 5890.	5184 Mg.	Star.	D_1 5896.	D_2 5890.	5184. Mg.
α CMi.	0·49	0·42	0·77	α Ari.	2·12	1·73	2·36
β Cas.	..	..	0·54	ε Peg.	..	..	2·10
α Per.	0·89	0·75	0·60	α Tau.	2·62	2·19	2·82
γ Cyg.	1·01	0·91	1·04	δ Oph.	2·83	2·57	2·59
Moon	1·04	0·84	1·65	β And.	2·65	2·39	2·54
α Aur.	1·21	1·02	1·59	α Ori.	2·83	2·67	3·06
β Gem.	1·49	1·24	2·11	α Sco.	3·23	2·93	2·45
α Boo.	1·50	1·25	2·16	β Peg.	..	..	3·09?

In figs. 4 and 5 are plotted the observed central and full absorptions † against the spectral classes or temperatures.

It is to be noted that *Na* has no maximum that is in agreement with theory. The curves 10^{-5} or 10^{-6} for the electron-pressure are in better agreement with the observations. The curve 10^{-4} requires the intensity constant nearly from K_2 to the later types, and this does not hold.

As to the sharp *Mg* triplet the earlier determinations by Menzel ‡ fix the maximum near K2–5, and this leads to the very small electron-pressure of 10^{-7}. We must remember here the blending effect mentioned in § 3. My observations based on the free-from-blends line 5184 give a very ill-defined maximum not earlier than K5, and this leads to a still smaller electron-pressure. But I am sure that this result is due to the rapid intensity growth of the titanium oxide band (near 5166), which, beginning at K_0, highly affects the intensity determinations in this region. For this reason the maximum determinations for the sharp *Mg* triplet, as very doubtful, are to be rejected.

On the other hand, the diffuse *Mg* triplet near 3830, as free from blending, is more suitable for the maximum determinations. Though the observations relate only to a few brighter stars we can surely fix its maximum not later than Ko (probably near G5). In any case, from Ko we have an evident decline. This leads to an electron-pressure between 10^{-4} and 10^{-5}. There are no other determinations for this triplet.

Though the detailed investigation of the dependence of the multiplets

* *Zs. f. Phys.*, **53**, 248, 1929.

† In the discussion in question it is more profitable to use the full absorption. In the light of modern representations it remains doubtful whether the change of central intensities along the spectral sequence characterises the change of atomic fractional concentration.

‡ *H.C.*, 258, 1927.

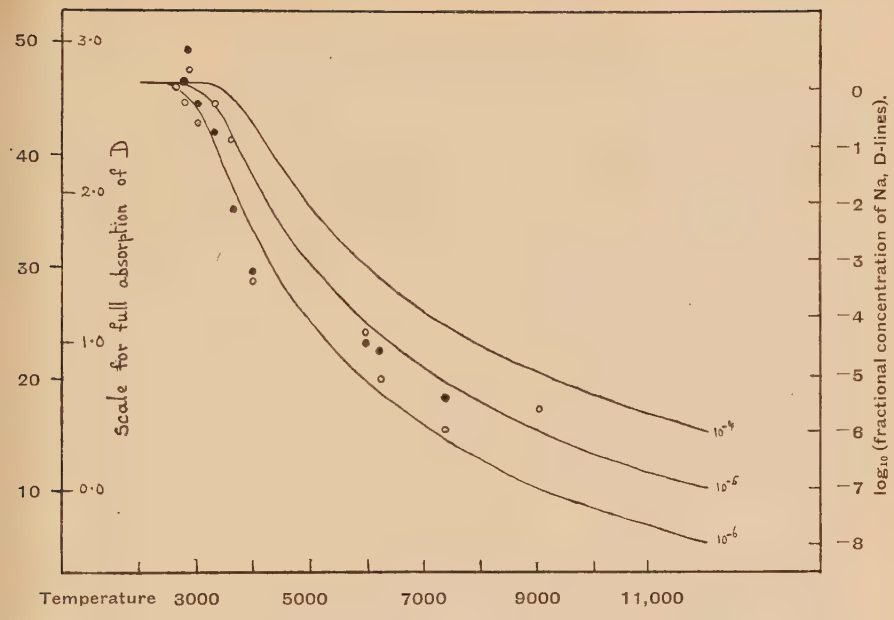


FIG. 4.—Fractional concentrations of *Na* (1^2S-1^2P) for the partial electron-pressures 10^{-4}, 10^{-5}, and 10^{-6}, and the observed D-lines absorptions.

○ –central absorption. ● –full absorption.

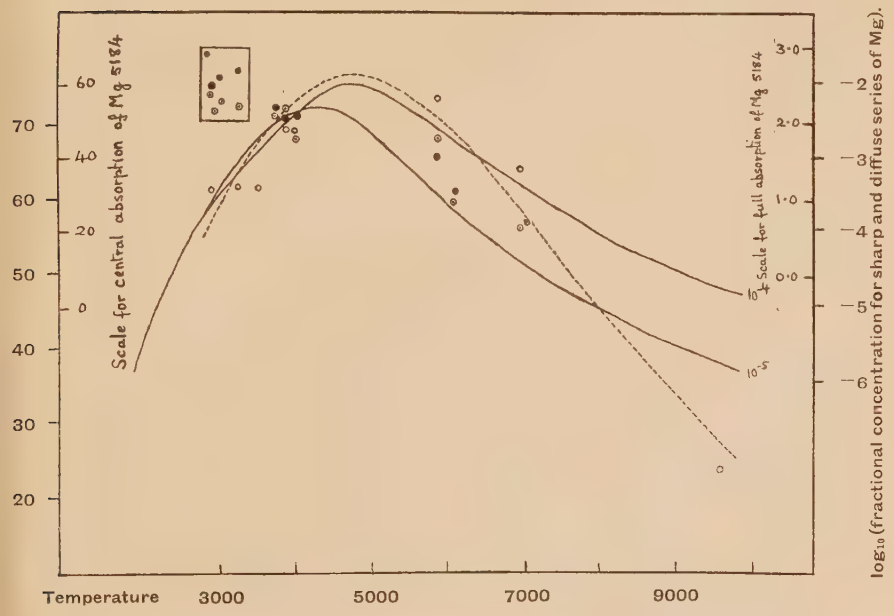

Scale for central absorption of Mg 5184
Scale for full absorption of Mg 5184
log10 (fractional concentration for sharp and diffuse series of Mg).
Temperature 3000 5000 7000 9000

FIG. 5.—Fractional concentration of *Mg* (1^3P-1^3S ; level 1^3P) for the partial electron-pressures 10^{-4} and 10^{-5} (full curves) and the observed line absorptions.

○ –central absorption for diffuse triplet near 3830 (broken curve).
⊙ –central absorption for line 5184 (sharp triplet).
● –full absorption for line 5184.

in question and many others upon the luminosity will be shortly discussed elsewhere, it may be noted here that in super-giants *Na* is stronger than in giants.

A peculiarity must be emphasised in α Cygni : *Na* lines in this star of type A_{2s} are nearly as strong as in stars of type F5.

To compare the observed data with theory we computed the curves representing the fractional concentration of atoms according to formulæ given by R. H. Fowler.* The curves as computed with right values of $u(T)$ and of the statistical weights have some advantage in comparison with those given in *M.N.*, **83**, 403. It is known that these formulæ have properly no relation to the intensity of lines. Generally it is assumed only that, other things being the same, the intensity of a given line varies always in the same sense as the concentration of the atoms in the reversing layer capable of absorbing this line. But the line intensity depends on the optical depth, and therefore on the coefficient of absorption and on the number of atoms in question. At present we cannot be sure that the absorption coefficient is constant for all the elements † and for all the states of excitation. For this reason the study of absorbing capacity of atoms of different elements as a function of the excitation state seems to be an urgent problem. For instance, the Russell-Adams phenomenon, as suggested by Menzel,‡ may be due not to the excess of atoms in higher excitation states in comparison with that given by thermodynamic equilibrium, but to the change of the atomic absorption coefficient with increasing excitation potential. As is known, determinations of maxima of spectrum lines interpreted in the light of Milne's revised theory allow us to derive the atomic absorption coefficient. Unfortunately, at present this method may be confidently employed only in few cases, since for the majority of the metallic and other narrow lines the determination of line-contours remains a doubtful operation. In addition, for the majority of lines with their maximum in the latest spectral classes, the blending effect, the uncertainty of the continuous background, and the growth of absorption bands such as titanium oxide are factors which severely hinder the accurate determination of the maximum.

If the theory is right in predictions of the fractional concentration of atoms, its comparison with observations allows us to give some of the information in question. This may be illustrated by a comparison of the sharp and diffuse *Mg* triplets, the potential excitation of which is the same. The theoretical curves of concentration for both triplets may be considered as identical. But the observed line intensities for the same temperature are highly different, as is evident from Tables II. and III. and fig. 5.§ Notwithstanding the difference in dispersion (1 mm. = 22 A. and 66 A. at 3830 and 5180 respectively), it seems that the cause of this phenomenon is not wholly of instrumental character. Two interpretations may be given : (1) the difference in absorbing capacity for these series, and (2) the difference of contrast of lines in the

* *Statistical Mechanics*, p. 371.

† Payne, *H.B.* 867, 1929.

‡ *P.A.S.P.*, **42**, 113, 1930.

§ The central absorptions for 3838 and 5184 are 66 and 26 respectively (α CMi.).

violet and red regions of the spectrum, the study of which deserves attention from both the theoretical and observational points of view. Another illustration consists in the comparison of *Na* and *Mg* (sharp series). Concentration of the order 10^{-5} and 10^{-4} corresponds to the same central absorption, viz. 20, for each of the elements in question at the dispersion used about F5. But to reach the absorption 50, which holds for both elements about K0, the concentration must be increased 10^5 and 10^2 times approximately for *Na* and *Mg* (sharp triplet, line 5184) respectively, provided the electron-pressure is the same. This is probably true also for the full line absorption.

It seems that such a comparison of the theoretical curves of atomic fractional concentrations with the observed intensity curves, and the last-named between themselves, may give some valuable information, in addition to the maxima determinations. Also, it is required to investigate in detail the instrumental effects.

Conclusions.—The principal preliminary results so far reached are as follows :—

The most striking feature of the multiplets in stellar spectra consists in a very small difference between the central absorption of lines within a multiplet. The contrast between the lines is much smaller than that observed in the laboratory emission spectra or predicted by the multiplets formulæ for emission, an analogy with the behaviour of the Balmer series in stellar spectra.

A constancy holds in the intensity ratio within a multiplet along the spectral sequence, and this is probably due to the small range of electron-pressure in the reversing layers for the atoms in question.

Within a spectrum class there are real differences in the intensity ratios of lines in individual stars.

A smaller contrast is suggested for the stronger lines within a multiplet.

The ratio of total line absorptions of H and K brings out a deviation from that predicted by the theory for the case of damping radiation.

The ratio of emission lines in H and K in certain late-type stars shows to some extent a closer approximation to that observed in the laboratory emission spectra, but the deviation remains large.

The current theoretical considerations are discussed, and it is suggested that on some rough but reasonable assumptions with respect to the ratio of line-absorption coefficients within a multiplet the observed facts may be accounted for by Eddington's model for absorption lines, provided the rôle of collision effect becomes important.

The observations of *Na* and *Mg* are discussed in the light of ionization theory and some suggestions are made with regard to the correlation between the observed intensity curves and the theoretical ones giving the atomic fractional concentrations.

I wish to express my thanks to Professor B. P. Gerasimovič for his interest in this work and for his useful advice.

Simeis,
1931 March.

A New Variable Star in Auriga.

By A. Stanley Williams.

R.A. = 7h 14m 10s. Decl. = + 36° 43′·5 (1855).
„ = 7 17 11. „ = + 36 38·6 (1900).

The variability of this star does not seem to have been hitherto noticed, since it is not in any of the lists of variables to which I have been able to refer, nor is it contained in Prager's *Katalog und Ephemeriden Veränderlichen Sterne für* 1931, which includes 4581 variable stars. Its variability was first noticed in 1921 November when comparing two photographs which had been taken about 20 years previously with a 4·4-inch portrait lens. Photographic plates, six in number, indicated a variation to the extent of nearly half a magnitude, the star being faint on two of them.

Visual observations were made during the winter of 1921 22 with a 6½-inch reflector. These soon confirmed the variability of the star, and showed also that it must be in a state of almost continual change of very short period. Unfortunately the observations of this winter are not very numerous or well distributed, as they were made chiefly in such intervals as were available between systematic work on other variable stars. Consequently there are seldom more than two or three observations on any one night, and often not more than one. I had intended remedying the deficiencies of the 1921–22 observations by means of a better series during the past winter, but this could not be done.

Under the circumstances and on looking through the early observations, it seemed worth while to reduce and discuss these incomplete observations and see what could be made of them. The result has been much better than was hoped for, since these observations show that the star is a rather interesting variable of the β-Lyræ type with the very short period of $13^h\ 40^m$.

Below are the positions for 1855 of the three comparison stars made use of, together with a (short) light-scale in steps and the corresponding magnitudes of the stars on the assumption that the star *a* is 9·8 mag. and that the value of a step is 0·05 mag., this last being probably very near the truth. The star *a* is BD. 36° 1618 (9·5 mag.).

Comparison Stars.

Star.	Position.		Light-scale.	Mag.
	h m s	° ′		
a	7 13 38	+36 46·7	30	9·80
d	7 14 11	+36 41·5	13	10·65
e	7 14 18	+36 41·3	11	10·75

The variable forms a little triangle with the stars *d* and *e*, and is usually considerably brighter than the other two. The comparison stars were not, however, very suitable, the interval between *a* and *d*

being too great for satisfactory estimation, even when the variable is near its maximum brightness, whilst there are several small stars close to *e*. Moreover, there are suspicions that either *d* or *e* may be slightly variable, to the extent of, perhaps, a tenth of a magnitude.

Table I. contains the particulars of the original observations. The times are geocentric Greenwich Mean Times. Column 5 gives the reduced brightness of the variable in steps of the light-scale, and column 6 the phase. The last-named has been computed from the elements

$$\text{Epoch} = \text{J.D. } 2423080{\cdot}400 + 0{\cdot}5694^{\text{d}}\,\text{E},$$

in which the epoch is an arbitrary one, chosen without reference to the actual maximum or minimum brightness of the variable.

TABLE I.

Observations.

1921.	G.M.T. h m	J.D.	Observations.	Brightness. s	Phase d
Dec. 13	9 53	2423037·412	*a* 9 v 9 *d*	20	0·286
22	11 21	3046·474	*a* 6 v 6 *d*	20	·238
23	10 23	3047·432	*a* 10 v 4 *d*	18	·057
23	10 49	·451	*a* 12 v 4 *d*, v 5 *e*	17	·076
23	12 19	·513	*a* 12 v 4 *d*, v 5 *e*	17	·138
25	8 43	3049·363	*a* 7 v 6 *d*, v 10 *e*	21	·280
25	12 21	·514	*a* 12 v 2 *d*, v 5 *e*	15·5	·431
25	12 49	·534	*a* 12 v 2 *d*, v 5 *e*	15·5	·451
25	13 14	·551	*a* 12 v 2 *d*, v 5 *e*	15·5	·468
25	15 09	·631	*a* 10 *v* 4 *d*, v 7 *e*	18	·548
29	7 32	3053·312	*a* 8 v 5 *d*, v 8 *e*	19	·245
31	10 19	3055·430	*a* 12 v 2 *d*, v 5 *e*	16·5	·084
31	11 11	·466	*a* 13 v, v 1 *d*, v 4 *e*	13·5	·120
31	12 14	·510	*a* 12 v 2 *d*, v 5 *e*	16·5	·164
31	14 35	·608	*a* 10 v 4 *d*, v 7 *e*	17·5	·262
1922.					
Jan. 3	7 54	3058·329	*a* 14 v 0 *d*, v 4 *e*	14	·136
3	8 49	·367	*a* 11 v 3 *d*, v 6 *e*	16·5	·174
3	10 35	·441	*a* 10 v 4 *d*, v 7 *e*	17·5	·248
4	8 15	3059·343	*a* 10 v 4 *d*	18	·011
17	8 09	3072·339	*a* 11 v, v 3 *d*	17	·480
17	9 05	·378	*a* 9 v, v 5 d, v 8 *e*	18·5	·519
17	9 42	·404	*a* 9 v, v 5 *d*, v 7 *e*	18	·545
17	12 18	·512	*a* 12 v 2 *d*, v 5 *e*	15·5	·084
21	9 13	3076·384	*a* 9 v 5 *d*, v 8 *e*	18·5	·539
21	10 02	·418	*a* 9 v 5 *d*, v 8 *e*	18·5	·573
21	12 37	·526	*a* 14 v 0 *d*, v 4 *e*	14	·112
25	8 13	3080·342	*a* 9 v 7 *d*, v 10 *e*	20	·511
25	9 16	·386	*a* 9 v 7 *d*, v 9 *e*	19	·555
26	9 21	3081·389	*a* 14 v, v 1 *d*, v 4 *e*	14·5	·420
Feb. 4	8 56	3090·372	*a* 10 v, v 6 *d*, v 8 *e*	19	·292
4	9 54	·412	*a* 10 v, v 5 *d*, v 8 *e*	18·5	·332
4	11 42	·487	*a* 14 v, v 2 *d*, v 4 *e*	15	·407
18	8 15	3104·344	*a* 8 v 8 *e*, v 6 *d*	19	·029
21	8 52	3107·369	*a* 9 v 8 *e*, v 6 *d*	19	·207
21	11 15	·469	*a* 10 v 6 *e*, v 4 *d*	17	·307
21	12 22	·515	*a* 12 v 4 *e*, v 2 *d*	15	·353
26	8 50	3112·368	*a* 14 v 1 *d*, v 3 *e*	14	·082

Table I.—*continued.*

1922.	G.M.T. h m	J.D.	Observations.	Brightness. s	Phase. d
Feb. 26	9 25	2423112·392	*a* 15 v, v 0 *d*, v 2 *e*	13	·106
26	11 07	·463	*a* 10 v, v 3 *d*, v 5 *e*	16	·177
26	12 33	·523	*a* 11 v, v 5 *d*, v 7 *e*	18	·237
26	13 40	·569	*a* 12 v, v 4 *d*, v 6 *e*	17	·283
Mar. 21	8 33	3135·356	*a* 8 v, v 6 d, v 8 *e*	19	·294
22	8 23	3136·349	*a* 12 v 4 *e*, v 2 *d*	15	·148
22	9 08	·380	*a* 14 v, v 2 *d*, v 4 *e*	15	·179
22	10 44	·447	*a* 11 v, v 5 *d*, v 7 *e*	18	·246
22	11 20	·472	*a* 10 v, v 7 *d*, v 8 *e*	19	·271
26	8 57	3140·373	*a* 10 v, v 3 *d*, v 5 *e*	17	·186
Apr. 4	8 40	3149·361	*a* 15 v, v 0 *d*, v 3 *e*	13·5	·064
17	10 06	3162·421	*a* 10 v 6 *e*, v 4 *d*	18	·027
19	9 54	3164·412	*a* 10 v, v 5 *d*, v 6 *e*	18	·310
Oct. 6	15 25	3334·642	*a* 12 v 8 *e*, v 7 *d*	19	·290
7	14 25	3335·601	*a* 16 v, v 4 *e*, v 2 *d*	14·5	·110

The observations in Table I., 52 in number, having been arranged in order of phase, were formed into 23 groups of 4 observations to a group with overlaps, two old observations being rejected and two fresh ones added to form each group. In this manner the number of points available for determining the mean light-curve was increased and the curve was also smoothed in the process. Before doing this, however, two of the observations of Table I. were rejected for discordance. The observation of Dec. 23, $12^h\ 19^m$ departs from the mean light-curve to the extent of about a tenth of a magnitude, not an abnormal amount in itself, but large in the case of a variable with a range of variation of no more than a quarter of a magnitude. The observation was one of the earliest ones, very likely made before the observer had acquired familiarity with the comparison stars and got into a settled habit of observing. The other rejected observation is that of April 4. Here the discordance amounts to a full tenth of a magnitude, but a slight readjustment of the adopted period or of the mean light-curve might easily reduce the apparent error. Nevertheless I am inclined to think that this discordance may represent a real fact, and that the minimum may have actually occurred, or at any rate begun, 20 or 30 minutes early on this occasion. Unfortunately there is no other observation of the same night which might have served as a check on this apparently discordant observation.

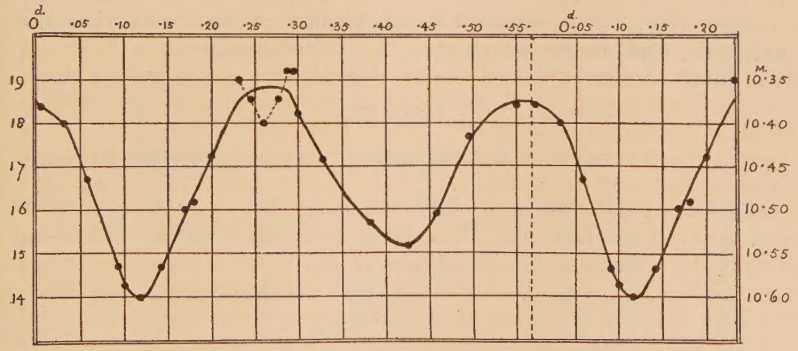
The grouped or mean observations are given below in Table II., and have also been reproduced in diagram form in the accompanying figure, which shows also the mean light-curve drawn through the points.

It will be seen that the agreement of the observations with the mean light-curve is quite satisfactory. There is a little scattering about the maximum following the principal minimum, but even here the greatest discordance barely amounts to half a tenth magnitude. As already mentioned, the times of the observations are geocentric times, but in view of the small number of observations and the little range in brightness of the variable it has not been thought worth while

TABLE II.

Mean Observations.

Phase.	Brightness.	Phase.	Brightness.
d	s	d	s
0·031	18	0·274	18·6
·061	16·7	·285	19·2
·089	14·7	·290	19·2
·103	14·2	·301	18·2
·119	14·0	·325	17·1
·142	14·7	·378	15·7
·166	16·0	·427	15·1
·179	16·1	·457	15·9
·202	17·2	·494	17·7
·232	19·0	·547	18·4
·244	18·6	·007	18·4
·257	18·0		



Light-curve of variable star in Auriga.

to reduce the times to the sun, particularly since no minimum or maximum has been directly observed on any one night. But the omission of this correction has probably resulted in a slight spreading of the observations. The mean light-curve as here derived presents some interesting peculiarities, and I hope that some other observer may be persuaded to make a better study of this star, so as to confirm, or otherwise, these peculiarities.

The epoch of the diagram is an arbitrary one, as already mentioned. Reduced to the time of the principal minimum, the following will be the concluded elements of variation of the star:—

$$\text{Principal Minimum} = \text{J.D. } 2423080{\cdot}515 + {}^{\text{d}}0{\cdot}5694\ \text{E.}$$

Assuming a magnitude of 10·35 for the maximum brightness of the star, that of the principal minimum is 10·60 and of the secondary minimum 10·55, the range of variation amounting to 0·25 mag.

Polvarth, St. Mawes, Cornwall :
1931 *March* 12.

Revised Elements of Variation.

Since the completion of the foregoing account, additional observations have been found, enabling a better determination to be made of the elements of variation of this star. There were 32 observations of its brightness made during the winter of 1922–3 ; 21 during that of 1923–4 ; and 23 during that of 1924–5. These observations have been reduced and plotted in the same manner as the former ones, and the times of a minimum fixed with the help of the mean light-curve. Also a good direct observation of the time of principal minimum was secured on the 18th March of the present year. It is likewise possible now to make use of the two early photographic observations referred to in the foregoing paper. The concluded revised elements of variation are :

$$\text{Heliocentric Principal Minimum} = \text{J.D. } 2423080{\cdot}520 + 0{\cdot}569287\ \text{d}\ \text{E}.$$

Table III., below, contains the particulars of the observations now available, with, in the last column, the differences O—C.

Table III.

E.	Date.	Observed Helioc. Minimum J.D.	Wt.	Computed Minimum.	O—C.
					d
−13,336	1901 Apr. 13	241 5488·502	ph.	·509	−0·007
−12,707	1902 Apr. 6	5846·517	ph.	·590	− ·073
0	1922 Jan. 25	242 3080·520	2	·520	·000
560	1922 Dec. 10	3399·340	1	·321	+ ·019
1,282	1924 Jan. 25	3810·331	1	·346	− ·015
1,861	1924 Dec. 20	4139·943	1	·963	− ·020
5,858	1931 Mar. 14	(6415·361)	F	·403	(− ·042)
5,865	1931 Mar. 18	6419·387 *	2	·388	− ·001

The first two observations are photographic, and the times are those of the middle of the exposures, 40^m and 60^m respectively. These exposures, especially the latter, are long for the rapid changes occurring near the time of minimum, and all that can be definitely said is that the variable was then at or near its minimum brightness. The observation on 1931 March 14 was a single, indifferent estimation making the star rather faint at the time (March 14, $20^h\ 39^m$ G.M.T.). If the observation had been made during the decrease in brightness the time of the minimum would come out as J.D. 2426415·421, in reasonable agreement

* 1931 March $18^d\ 21^h\ 17^m$ G.M.T.

with the last observation. During the 30 years covered by the observations the variable, considered as a binary system, performed 19,201 revolutions without any definite indication of a change in the length of the period.

The light-curve bears considerable resemblance to A. Roberts's light-curve of V Puppis, as shown in the figure on p. 123 of Miss Clerke's *System of the Stars* (2nd ed., 1905), so that some of the conclusions that have been drawn concerning that star may apply more or less to the present variable. The orbit plane would appear to be more inclined to the line of sight in the case of the latter, however. The additional observations now available tend to remove some of the irregularities shown by the present mean light-curve. The principal minimum becomes slightly narrower and the two maxima a little broader and of equal height, whilst the secondary minimum becomes more symmetrical.

1931 *March* 27.

Errata.

P. 472, line 4, *for* "$n = \frac{3}{2}$"-solutions *read* "centrally-condensed $n = \frac{3}{2}$"-solutions.

P. 480, line 32 from top, for *same* read *some.*

P. 498, formula (31), *for* $y = \frac{\kappa\delta R^2}{3}$ *read* $4y = \frac{\kappa\delta R^2}{3}$.

P. 498, formula (33) *should read*

$$4\kappa p R^2 x = \left(\frac{dx}{d\tau}\right)^2 - (2x - 1)^2 + 1 - 16xy.$$